SMALL SAMPLE POWER OF SOME

ASYMPTOTICALLY OPTIMUM RANK TESTS*

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Errata for T. R. # 46

P. 1, Line 11, "unkown" should be "unknown"

p. 1, Line 14, ... uniformly most powerful among unbiased procedures."

p. 5, Line 3, $\frac{1}{2}e^{+\sqrt{2}(x-\theta)}$, $x \leq \theta$

p. 6, Line 13, " $|X_1$," should be "$|X_1|$,"

p. 6, Line 106, $3/\pi = .955$

p. 6, Line 16, "these test statistics are ..."

p. 7, Line 7, $h_n(u) = \begin{cases} -\sqrt{2} \text{ if } u < 0 \\ +\sqrt{2} \text{ if } u \geq 0 \end{cases}$

p. 7, Line 9, $S_n^* = \sqrt{2} \sum_{X_i > 0} 1$

p. 27, Line W15 (logistic) is upper unbroken line
S15 (logistic) is lower unbroken line
W15 (normal) is upper broken line
S15 (normal) is lower broken line

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TABLE OF CONTENTS

INTRODUCTION

A well known problem in statistical hypothesis testing is that of deciding whether a sample $(X_1,\ldots,X_n)$ of observations came from a population with a specified mean, θ_o. For example, $(X_1,\ldots,X_n)$ might be n independent observations of the number of units produced per hour, where production was carried out by a new method. Suppose that the old method is known to produce an average of θ_o units per hour with variance σ^2. The problem of deciding whether or not to replace the old method with the new can be described as a problem of deciding whether the mean output of the new process is greater than θ_o.

The classical approach to this problem is based on $t' = \dfrac{(\bar{X} - \theta_o)}{\sigma}$ when σ is known, and on $t = \dfrac{(\bar{X} - \theta_o)n^{\frac{1}{2}}}{s}$ when σ is unkown, where $\bar{X}$ and s^2 are the mean and variance of the observations. In both cases, if the data are from a normal population, the one-sided upper tail procedure is a best procedure in the sense it is uniformly most powerful. The power of a test is its ability, in terms of probability, to detect departures from the null hypothesis; in the example above, the ability to detect when the new mean is greater than θ_o. Most powerful denotes most powerful of all tests for the situation; uniformly most powerful signifies most powerful for all possible means greater than θ_o. When σ is known, the power depends on $(\theta-\theta_o)^2$, where θ is the location parameter (mean) of the alternative hypothesis. The power in this case can easily be computed from tables of the standard normal distribution. When σ is unknown, the power depends on $(\theta-\theta_o)^2/\sigma^2$, and can be calculated from tables of the non-central t-distribution. The power in both of these instances will be correct provided the data are from a normal population, and for large enough samples it will be approximately correct for any population with finite variance.

The question arises as to the appropriateness of this conventional approach when the data is not from a normal population. For large samples it is reasonable to use the above methods whenever the population is known to be approximately normal. In the case where there is moderate or small sample size and it is not known that the assumption of normality is justified, alternative methods are desired. If the basic family of distributions has a parametric form (but not necessarily normal), one may be able to derive a test based on that family. However, in many cases the experimenter does not know the form of the basic family and needs techniques which are applicable for wider classes of distributions. A technique with this property is called non-parametric. A technique designed for a parametric class of distributions is called parametric.

This paper considers a class of non-parametric procedures known as rank tests. These tests assume, for the one sample problem, only continuity and symmetry of the parent population and independence of the observations. It is generally true that these tests have less power for a specified distributional form than the corresponding best parametric test designed specifically for the given family. Whether rank tests have other properities to make them desirable, e.g., relatively stable power over a wide class of alternative distributions, is partially what we explore here.

Since there are a number of rank tests, a problem of choice arises. One would like to use a test with high power for a wide range of alternatives, and possibly with its highest power for a certain distribution about which there is concern. Hájek has presented a theory which specifies an optimum rank test for large samples. There is a question as to whether this theory applies to small samples. This question and the problem of choice of test are dealt with in this paper.

The chief purpose of this paper is to present some tables of power functions for several rank tests and population distributions. These power functions were obtained by extensive Monte Carlo methods on a digital computer. The tables are for the .05 and .10 significance levels and are designed to facilitate power comparisons. The null hypothesis, the parent distributions, and the tests are presented in chapter I. The tables, their accuracy and calculation are discussed in chapter II. Some uses of the tables are suggested in chapter III, where the questions of Hájek's theory and choice of rank test are addressed.

I PRELIMINARY REMARKS

1.1 The Hypothesis of Interest

Let $f(x)$ be a symmetric density, i.e.,

$$f(x) = f(-x), \quad \infty < x < \infty.$$

Consider a random sample $X_1, X_2, \ldots, X_n$ where

$$P(X_i \leq x|\theta) = \int_{-\infty}^{x} f(t-\theta)dt.$$

We are interested in the hypothesis that $\theta = 0$ against the alternative $\theta > 0$. The function $f(x-\theta)$ defines a family of densities indexed on θ. The family of probability distributions corresponding to this family of densities is called the parent distribution family. The scale parameter σ is incorporated in f; a change in σ changes the family of f's.

This hypothesis may be interpreted more simply as follows. Given n independent observations $X_1, \ldots X_n$ from a distribution with symmetric distribution function $F(x-\theta)$, we wish to test the hypothesis $\theta = 0$ against the alternative $\theta > 0$. The situation can easily be modified to consider $\theta = \theta_o$ (vs) $\theta > \theta_o$ or $\theta = \theta_o$ (vs) $\theta < \theta_o$.

1.2 The Parent Distributions of Interest

Three parent translation families of distributions indexed on $\theta \geqq 0$ are considered throughout this paper. They are:

i) the normal translation family with density

(1.2.1) $$f_N(t;\theta) = (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}(t-\theta)^2}, \quad -\infty < t < \infty,$$

and distribution function

(1.2.2) $$F_N(x;\theta) = \int_{-\infty}^{x} f_N(t;\theta)dt, \quad -\infty < x < \infty$$

ii) the double exponential translation family of Laplace with density

$$(1.2.3) \qquad f_D(t;\theta) = \frac{1}{\sqrt{2}} e^{-\sqrt{2}|t-\theta|}, \quad -\infty < t < \infty$$

and distribution function

$$(1.2.4) \qquad F_D(x;\theta) = \begin{cases} \frac{1}{2}e^{-\sqrt{2}(x-\theta)}, & x \leq \theta \\ 1 - \frac{1}{2}e^{-\sqrt{2}(x-\theta)}, & x \geq \theta \end{cases}$$

iii) the logistic translation family with density

$$(1.2.5) \qquad f_L(t;\theta) = \frac{\pi}{\sqrt{3}} e^{-\frac{\pi}{\sqrt{3}}(t-\theta)} (1+e^{-\frac{\pi}{\sqrt{3}}(t-\theta)})^{-2}, \quad -\infty < t < \infty$$

and distribution function

$$(1.2.6) \qquad F_L(x;\theta) = \frac{1}{1 + e^{-\frac{\pi}{\sqrt{3}}(x-\theta)}}, \quad -\infty < x < \infty.$$

All three of these families are standardized in the sense that each member of each family has mean θ and variance one. The densities and distribution functions of these families for $\theta = 0$ are compared graphically in figures 1.1 and 1.2. With respect to terminology, the double exponential of Laplace mentioned above should not be confused with the double exponential distribution of Gumbel which has density

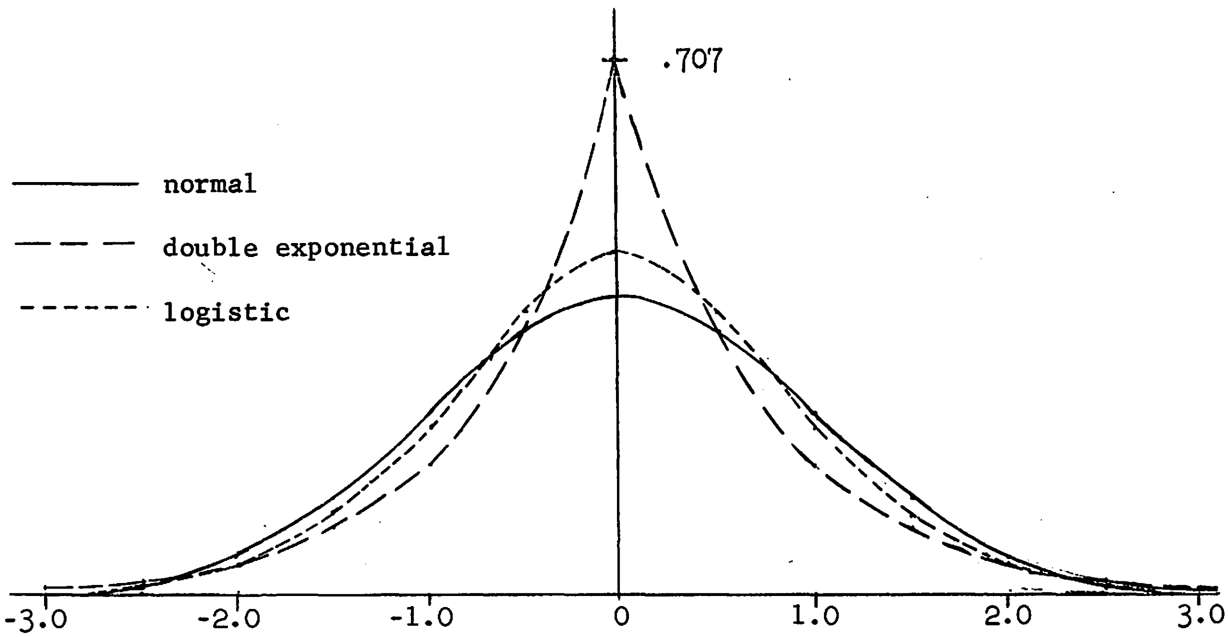
$$-\log_{10}[1-\exp(-\exp(-x))], \quad \infty < x < \infty.$$

F shall be used to refer to the distribution function of a general symmetric continuous distribution with mean zero. f will denote the density of F; $f(x;\theta)$ will denote this density shifted right by θ. F_k, f_k, $f_k(x;\theta)$, $k = N,D$, or L are defined similarly with reference to the above families. That is $F_N(x) = F_N(x;0)$, where $F_N(x;\theta)$ is defined in (1.2.2); and $f_N(x) = f_N(x;\theta)$, where $f_N(x;\theta)$ is defined in (1.2.1).

1.3 The Tests

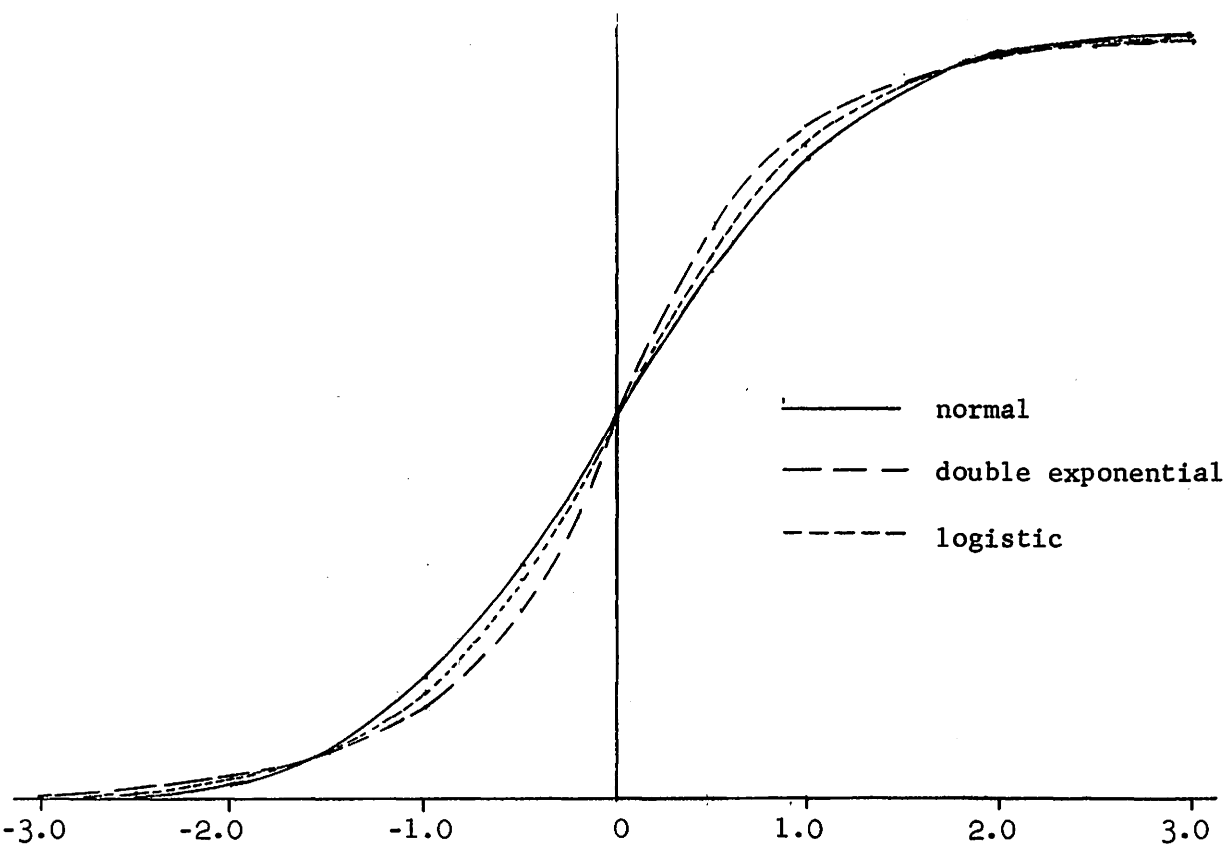
MOTIVATION AND DESCRIPTION

Suppose an experimenter has independent observations $X_1,\ldots,X_n$ from a population with distribution function $F(x-\theta)$. He is interested in testing the hypothesis of section 1.1, i.e., the null hypothesis $\theta = 0$ versus the alternative hypothesis $\theta < 0$.



Standardized densities

Figure 1.1



Standardized Distribution Functions

Figure 1.2

Under the null hypothesis the observations are equally likely to be positive or negative. Further, because of the assumption of symmetry, if the observations are ordered according to absolute value, the positive observations should be well dispersed among the negative observations. This dispersion can be tested to see if it meets the requirements of the null hypothesis. First replace the observations by signed ranks, i.e., replace the smallest observation in absolute value by +1 if it is positive and -1 if it is negative; replace the second smallest observation in absolute value by +2 if it is positive and by -2 if it is negative, etc. Then sum the positive ranks. On the average we can expect this sum to be $n(n+1)/4$; if it is much greater we reject the null hypothesis. This is the Wilcoxon signed rank test.

Symbolically, let R_i be the rank of $|X_i|$ in the sequence $|X_1, \ldots, |X_n|$ rearranged according to ascending order of magnitude and let

$$W = \sum_{X_i > 0} R_i$$

and reject $\theta = 0$ if $W > K$. This test has certain optimum properties for large n. For example its asymptotic efficiency with respect to Student's t for the normal translation family is $3/\pi = .995$. Lehmann discusses some of its optimum properties in [7].

A natural extension of this test is to consider sums

$$(1.3.1) \qquad H_n = \sum_{x_i > 0} h_n\left(\frac{R_i}{n+1}\right).$$

There is considerable theoretical basis for choosing functions h. Hájek [4] has shown that if

$$h(u) = -\frac{f'(F^{-1}(\frac{1}{2}+\frac{1}{2}u))}{f(F^{-1}(\frac{1}{2}+\frac{1}{2}u))}$$

then the test $H_n > K$ provides an asymptotically locally most powerful test of $\theta = 0$ against $\theta > 0$, provided f meets certain regularity conditions. These tests are also asymptotically normally distributed and asymptotically efficient.

Now if $F = F_N$, then

$$h_n(u) = \Phi^{-1}(\tfrac{1}{2} + \tfrac{1}{2}u)$$

and

$$V_n = \sum_{x_i > 0} \Phi^{-1}\left(\tfrac{1}{2} + \frac{R_i}{2(n+1)}\right)$$

is called the van der Waerden test (for one sample). If $F = F_D$, then

$$h_n(u) = \begin{cases} 0 \text{ if } u < 0 \\ 1 \text{ if } u \geq 0 \end{cases}$$

and

$$S_n^* = \sum_{x_i > 0} 1/n+1$$

or equivalently

$$S_n = \sum_{x_i > 0} 1$$

is called the sign test. If $F = F_L$, then

$$h_n(u) = u$$

and

$$W_n^* = \sum_{x_i > 0} R_i/n+1$$

or equivalently

$$W_n = \sum_{x_i > 0} R_i$$

is the Wilcoxon test. V_n, S_n, and W_n are asymptotically locally most powerful for the normal, double exponential, and logistic translation families respectively, since these distributions meet the regularity conditions of Hájek. V_n (one sample case) is conspicuous by its absence in the literature.

Note that h assigns a certain weight to a rank in a manner optimum for the family of the distribution from which the observation is taken. Figure 1.3 compares these assignments for the three translation families of interest. This figure indicates the double exponential translation family has a property that may be described by saying that large positive observations are no more indicative of $\theta > 0$ than small positive observations. The normal translation family, however, is quite different in this regard. The function h for normal gives the more extreme positive values considerably more weight than small positive values of X. This difference corresponds to the differences of distribution of mass in the two families (see figures 1.1 and 1.2).

The logistic translation family, which is, in a sense, between the normal and double exponential, has a weighting function h which in the same sense is between the weighting functions given by F_N and F_D.

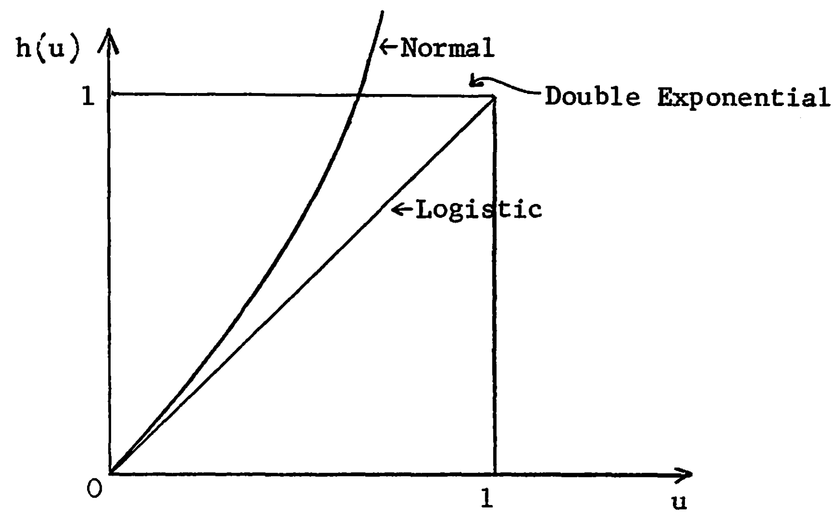


Figure 1.3

It should be mentioned that the assumption of symmetry is made so that it is equally likely that the j^{th} smallest absolute value has a positive or negative sign. In this case the null distributions of V_n and W_n are known. S_n does not depend on the ranking of the absolute values, but only on the signs of the observations. So the requirement of symmetry can be replaced for S_n by the lesser requirement that θ be the median.

Also the null distributions of all three of V_n, S_n, and W_n do not depend on the observations having equal variances. This fact is very important in the case where the observations are made under differing conditions. However, the assumption of equal variances was made in section 1.1 and will be carried throughout the paper. This is because the non-null distributions do depend on the variances and assumption of equality makes power studies more feasible.

The significance level α of a test is the conditional probability that the test statistic H_n will fall in the critical region CR given that the null hypothesis is true:

$$\alpha = \Pr(H_n \epsilon CR | \text{null hypo true}) .$$

In practice the critical resion is determined by specifying α (and the null hypothesis). The power, p, of a test is the conditional probability that the test statistic H_n falls in the critical region given that a specified alternative hypothesis is true:

$$p = \Pr(H_n \epsilon CR | \text{alt hypo true}) .$$

Once the critical region has been established by specifying α it is desirable to know p for various alternatives in order to evaluate the test procedure and the results of the test. To obtain the power of the tests described above in the situation of section 1.1 it is necessary to specify an alternative family and set of shift parameters θ. It is the purpose of this paper to present estimates of the power of V_n, S_n, and W_n for various alternatives and small sample sizes.

DISTRIBUTION AND CRITICAL VALUES OF THE TESTS

As mentioned before $H_n = \sum_{x_i > 0} h(\frac{R_i}{n+1})$ is asymptotically normally distributed under certain conditions on h and F. V_n, S_n, and W_n are all normally distributed in the limit. [4]

S_n is distributed as binomial; so its null and non-null distributions are relatively easy to calculate. The critical regions for S_n for $n = 2(2)10,15$, $\alpha = .05,.10$ are given in Table IV.

To the best knowledge of this writer, neither the null distribution nor the critical values for the one sample V_n statistic have been published. The exact critical regions of V_n for $n = 2(2)10$, $\alpha = .05, .10$ were calculated and are presented in Table IV. The ranks yielding the critical values are also given; this was thought advisable because the values often differ only in the second or third decimal place and hence knowledge of the critical ranking can save tedious calculations. Table IV also contains an approximation to the critical regions for $n = 15$. This approximation was used in computing the empirical power functions and appears to be quite close. The reader is referred to the appendix for further details.

Table V contains values of $\Phi^{-1}(\frac{1}{2} + \frac{k}{2(n+1)})$ for $n = 2(2)\ 10, 15$, $k = 1(1)n$ for computing W_n.

II DISCUSSION OF PRINCIPAL TABLES

2.1 Notation

The parameter values used throughout the tables of empirical power functions are sample size $n = 2(2)10,15$; shift parameter $\theta = 0(.25)1(.5)3(1)4$; significance levels $\alpha = .05, .10$. The parent distributions are the normal, double exponential, and logistic mentioned in section 1.2. These are abbreviated NORM, DE, and LOG respectively in the tables. The tests are V_n, S_n, W_n described in section 1.3; in the tables these are denoted VDW, SIGN, and WIL respectively, with the sample size n denoted at the top of the page. $H_n(\alpha)$ symbolizes the critical region for statistic H_n at significance level α. The uniform distribution on the interval (0,1) will be denoted U(0,1).

2.2 Description

TABLE I

Table I is the main table to be presented by this paper. It gives Monte Carlo estimates of

$$(2.2.1) \qquad \int_{H_n(\alpha)} \cdots \int f_k(t_1;\theta) f_k(t_2;\theta) \cdots f_k(t_n;\theta)\, dt_1 \cdots dt_n$$

where $k = N,D,L$ and $H_n = V_n, S_n, W_n$. These estimates are based on 42,500 sample points. All integrals for V_6 have been omitted from the table; error entered their calculation and it was found necessary to omit them. The integral (2.2.1) is the power of the test H_n at significance level α for the alternative distribution with density f_k and shift parameter θ.

TABLE II

Table II is similar to Table I except that the estimates are based on 42,000 sample points. This table is designed for use with a possible sequential procedure suggested on p. 14 .

TABLE III

Table III is also similar to Table I except that all estimates are based on 500 sample points. For all practical purposes it is statistically independent of Table II. This table is also for use with the sequential procedure.

FORMAT OF TABLES I - III

Table 2.0 represents a typical entry from Tables I - III. Specifically, it is from Table I, p. 39 . In the tables themselves, the sample size is at the top of the page; the distribution types are indicated just below the sample size. The significance levels and tests are listed on the left hand side of each page.

SAMPLE SIZE = 4							
		NORM	DE	LOG	NORM	DE	LOG
			$\theta = 0$			$\theta = .25$	
	VDW	.050	.050	.050	.104	.144	.113
α=.05	SIGN	.050	.050	.050	.104	.144	.113
	WIL	.050	.050	.050	.105	.145	.114
	VDW	.100	.100	.100	.197	.249	.210
α=.10	SIGN	.099	.099	.099	.180	.235	.193
	WIL	.101	.101	.101	.198	.250	.211

TABLE 2.0

Note there are four blocks of numbers in Table 2.0; each block corresponds to a shift parameter θ and a significance level α. For example,

.104	.144	.113
.104	.144	.113
.105	.145	.114

corresponds to $\theta = .25$ and $\alpha = .05$ (for $n = 4$). Now, within a block, each row corresponds to a test (labeled on the left hand side of the page) and each column corresponds to a parent distribution (labeled at the top of the page). Thus, for example, .145 is the power of WIL (V_n) for $n = 4$, $\theta = .25$, $\alpha = .05$ when the observations are from a double exponential distribution with variance one; .250 is the power when α is raised to .10.

The significance level, i.e. the power when $\theta = 0$, of the sign test (SIGN) for $n = 4$ for the logistic distribution (LOG) is .099, as it is for the other two distributions (NORM and DE).

2.3 Accuracy of Tables

All estimates are believed to be within .005 of true value; this belief is based to some extent on confidence intervals and to some extent on the stability of the estimates and the error observed in the known integrals that were estimated as checks.

Confidence Interval Approach

For single integrals to be estimated, what we have is a simple binomial problem. Let

$$p = \int_{H(\alpha)} f(\underset{\sim}{x})\, d\underset{\sim}{x} = P(H \epsilon H(\alpha) \mid \text{alt hypo})$$

be the integral we are trying to estimate. Of course $0 \leq p \leq 1$. Now define $Z_i = 1$ if $H \epsilon H(\alpha)$ and $= 0$ if $H \notin H(\alpha)$. Then

$$P(Z_i=1) = p$$

$$P(Z_i=0) = 1-p$$

Now if N observations are taken on Z_i, then $\sum_{i=1}^{N} Z_i$ is binomial (N,p), and $\sum_{i=1}^{N} Z_i/N$ is an unbiased estimate of p.

Let $\hat{p} = \sum_{i=1}^{N} Z_i/N$ and $\text{var}\ \hat{p} = \sigma^2$ and $q = 1-p$. For large N, $\sum_{i=1}^{N} Z_i$ is approximately $N(Np,Npq)$ and $\hat{p}$ is approximately $N(p,pq/N)$. Now by placing bounds on p, one can decide the accuracy and confidence we need in estimating p and solve for an upper bound on N, the number of sample points necessary. Or given N, bounds can be found for the accuracy and confidence of the estimates.

For example, if nothing is known about p one can at least say $p \leq \frac{1}{2}$ or $q \leq \frac{1}{2}$, so that $pq \leq \frac{1}{4}$ or $\text{var}\ \hat{p} \leq \frac{1}{4}N$. Suppose it is desired to be within $\pm .004$ of p with pr .95. This means it is necessary to be within 2σ of the mean, i.e., necessary for $2\sigma \leq .004$ or $\sigma \leq .002$ or $1/\ 4N \leq .002$ or $N \geqq 62{,}500$. That is 62,500 sample points will put $\hat{p}$ within $\pm$ .004 of p with pr .95, no matter what p is. To get accuracy $\pm$ .0004 with pr .95 in this case one would need $N \geqq 6{,}250.000$, i.e., 100 times as many points, whereas to get accuracy $\pm$ .04 only 625 sample points are needed.

However, if it is known that $p \geqq .9$ or $p \leqq .10$, then $pq \leqq .09$. To get within $\pm$ .004 with pr .95 would then require $2\sigma \leqq .004$ or $.09/N \leqq .002$ or $N \geqq 22{,}500$.

Or suppose we have $N = 42{,}500$ and we know only $p \geqq \frac{1}{2}$ or $q \geqq \frac{1}{2}$. Then $\text{var}\ \hat{p} \leqq \frac{pq}{n} = .588\times10^{-5}$ or $\sigma \leqq .0077$. Then it is possible to say, for example, $p \in (\hat{p}-.015,\ \hat{p}+.015)$ with probability .95.

Table 2.1 gives values of N for various bounds on p, degrees of accuracy, and confidence coefficients.

It is possible to carry out a sort of sequential procedure here. Five hundred sample points will put p within $\pm$ .04 of $\hat{p}$ with pr .94. This initial estimate, called $\hat{p}_1$, could be used to place a bound on p and with this bound the above procedures followed. A model would have to be set up conditioning of $\hat{p}_1$. Either the final N, the interval, or the confidence coefficient could be the final random variable. Tables II and III are provided for this purpose.

TABLE 2.1

Bound on P	Conf. Coeff.	Upper bound on N required for absolute error of $\pm.01$	$\pm.004$	$\pm.002$
$\geq .99$ or	.954	400	2500	9900
$\leq .01$	.998	900	5900	20000
	.682	475	3025	
$\geq .95$	.900	1325	8250	
or	.954	1900	12100	47500
$\leq .05$	.988	3000	19600	
	.998	4360	28100	
	.682	900	5625	
	.900	2400	18300	
$\geq .90$	.954	3600	22500	90000
or	.988	5600	35000	
$\leq .10$	.998	8100	53200	
	.682	2100	13200	
$\geq .7$	.900	5700	36500	
or	.954	8400	52500	210000
$\leq .3$	.988	13200	85500	
	.998	19200	130000	
	.682	2500	15700	
$\geq .5$	.900	6750	43400	
or	.954	10000	62500	250000
$\leq .5$	.988	15700	98000	
	.998	23000	148000	

Thus far only the estimation of a single integral has been considered. Another problem is what to do about the estimation of several integrals, of a large segment of the tables. One thing that can be said is that there is a high degree of dependency among the integrals of Table I. For example, the points used in estimating for $n = 2$ were also used for $n = 4$; and those used for $n = 4$ were also used for $n = 6$;

and so on. Moreover the parent populations are highly dependent, with the samples from each of the populations being calculated from a single sample from U(0,1). So it can be said roughly that if some of the integrals are accurate then the rest should also tend to be accurate. More precise statements could perhaps be made with the aid of, for example, the theorem on p. 89 of [1]; this has not been done here however.

Convergence and Error Checks

Results for $\theta = 0$ and the power of S_n provide possible checks on accuracy. Other partial checks are comparisons with Klotz's tables of power of W_n for normal alternatives [5]. The case of $\theta = 0$ provides the easiest check since this just corresponds to the significance level; these alone, however, are not sufficient because they are only for p small. The power of S_n provides checks for all ranges of p. The only trouble is that there are no tables of the binomial published that will give three places of accuracy where it is wanted here [2]; thus much manual calculation is necessary.

All experimental significance levels ($\theta = 0$) are within $\pm$ .003 of expected value. Other selected checks for S_n in Table I are presented in Table 2.2. Table 2.3 presents some comparisons of the results in Table I with some exact calculations by Klotz. Although direct comparison is not possible because of the difference in significance levels, some parallelism exists and is helpful.

Figures 2.1 through 2.7 display the convergence of some known and

TABLE 2.2

TABLE OF POWER SIGN TEST

N	θ	α	DIST	EXACT	ESTIMATE	ABSOLUTE ERROR
2	4.00	.10	NORM	.400	.401	.001
4	.50	.05	LOG	.206	.208	.002
4	1.00	.10	NORM	.558	.561	.003
4	4.00	.05	NORM	.800	.804	.004
6	.25	.10	LOG	.233	.234	.001
6	.75	.05	NORM	.352	.355	.003
6	1.00	.10	DE	.803	.804	.001
6	2.00	.05	NORM	.916	.916	.000
6	2.50	.05	NORM	.977	.977	.000
8	.50	.05	NORM	.278	.280	.002
8	.50	.10	NORM	.412	.412	.000
8	1.00	.10	DE	.860	.860	.000
10	.50	.05	DE	.507	.509	.002
10	.25	.05	LOG	.172	.173	.001
10	.75	.10	NORM	.686	.691	.005
15	.25	.10	LOG	.331	.329	.002
15	.50	.05	NORM	.439	.437	.002
15	.50	.10	NORM	.580	.578	.002
15	.50	.05	LOG	.509	.508	.001

TABLE 2.3

POWER OF W_n FOR NORMAL ALTERNATIVES*

		θ						
N	α	.25	.50	.75	1.00	1.5	2.0	2.5
6	.04688 (.050)	.1248 (.135)	.2630 (.274)	.4503 (.463)	.6464 (.658)	.9114 (.916)	.9887 (.989)	.9992 (.999)
8	.05469 (.050)	.1672 (.158)	.3689 (.346)	.6139 (.592)	.8180 (.796)	.9831 (.980)	.9995 (.999)	1.0000 (1.000)
10	.05273 (.050)	.1844 (.178)	.4274 (.414)	.7013 (.692)	.8914 (.887)	.9957 (.995)	1.0000 (1.000)	
10	.09668 (.100)	.2862 (.290)	.5669 (.575)	.8153 (.824)	.9476 (.951)	.9989 (.999)	1.0000 (1.000)	

*Four digit numbers are the exact calculations of Klotz. The three digit numbers in parenthesis are from Table I.

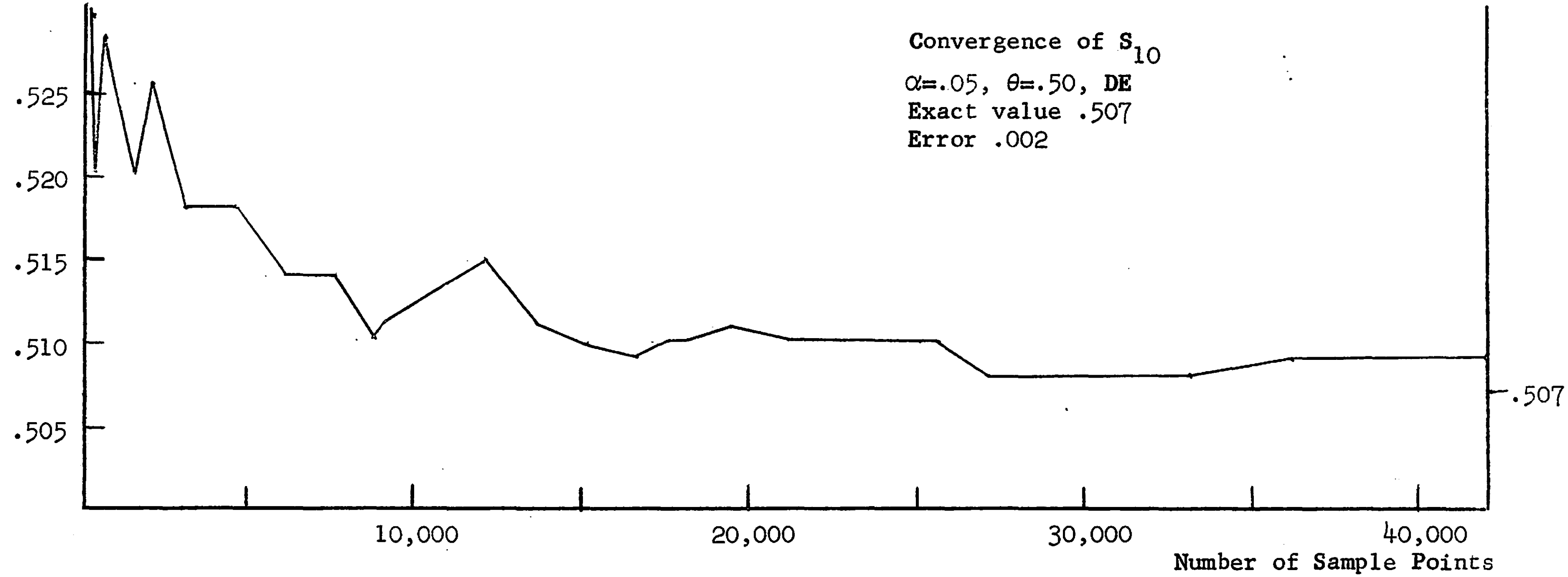


Figure 2.1

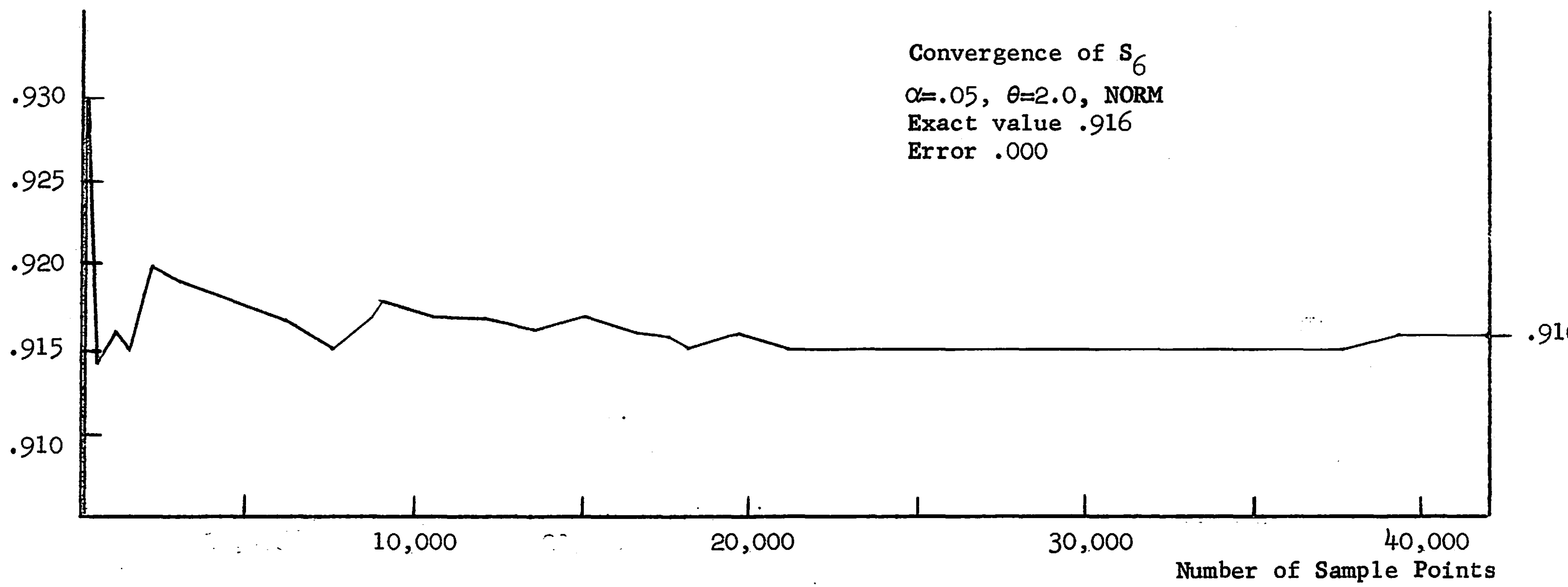


Figure 2.2

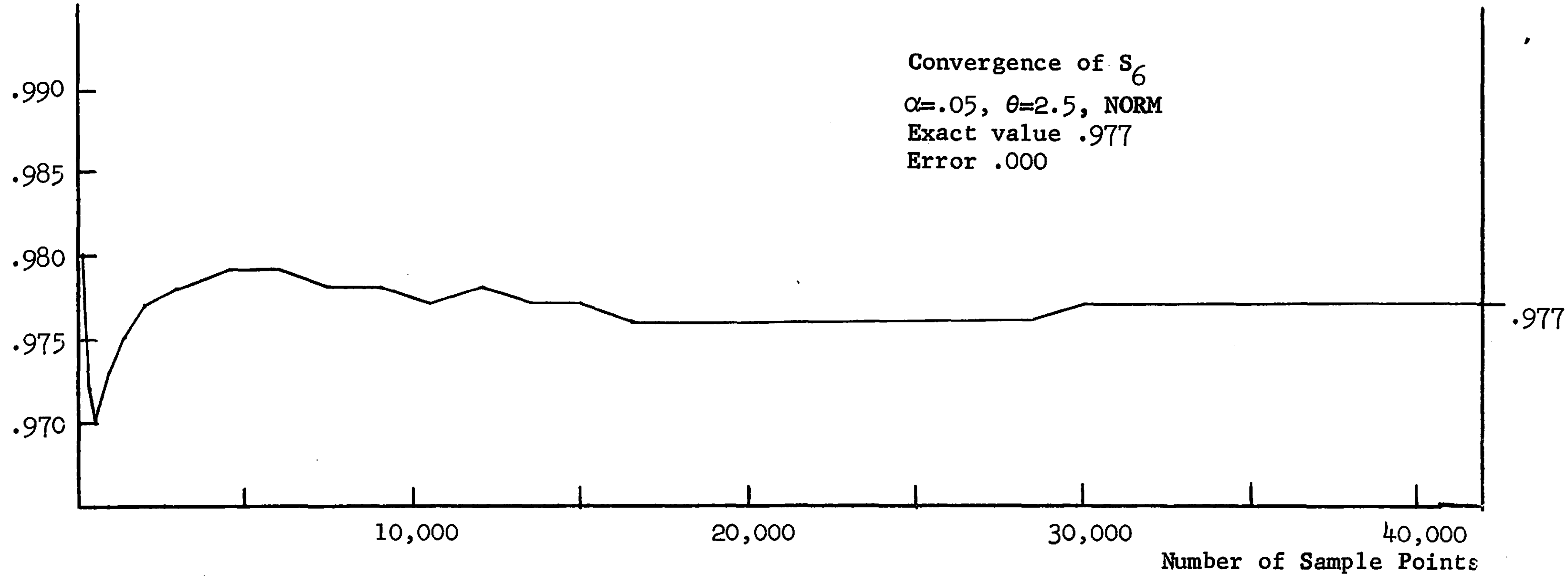


Figure 2.3

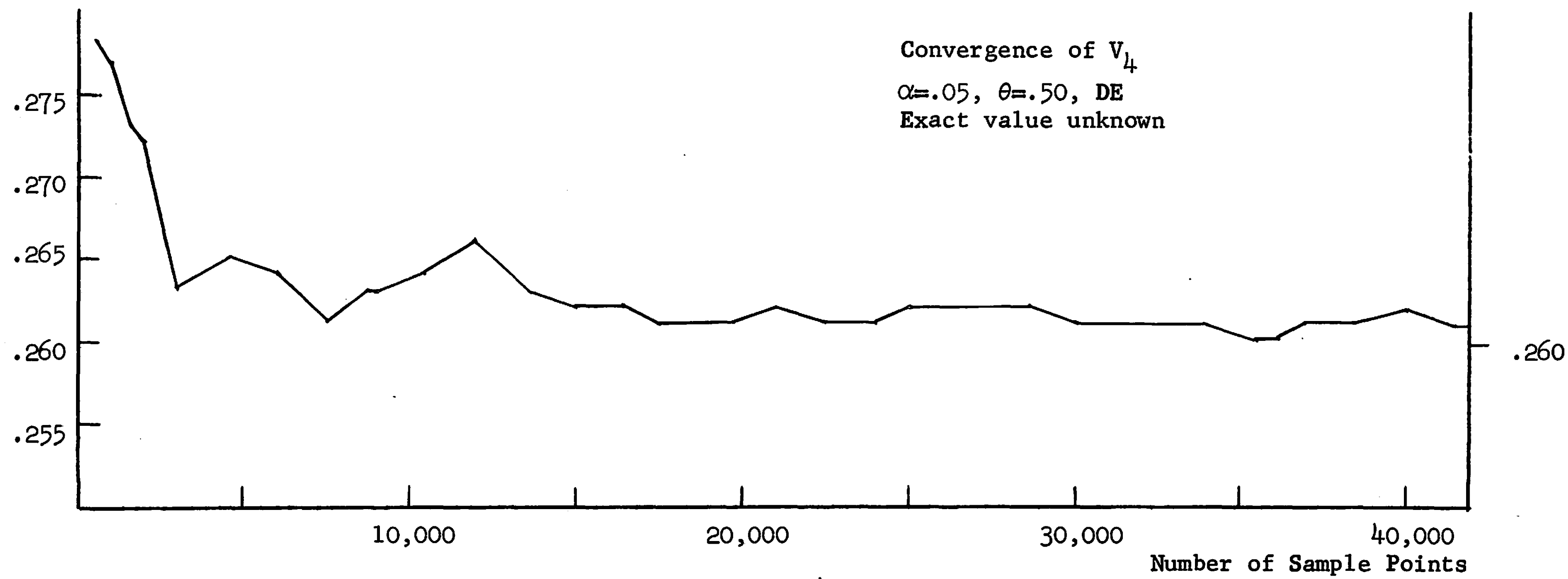


Figure 2.4

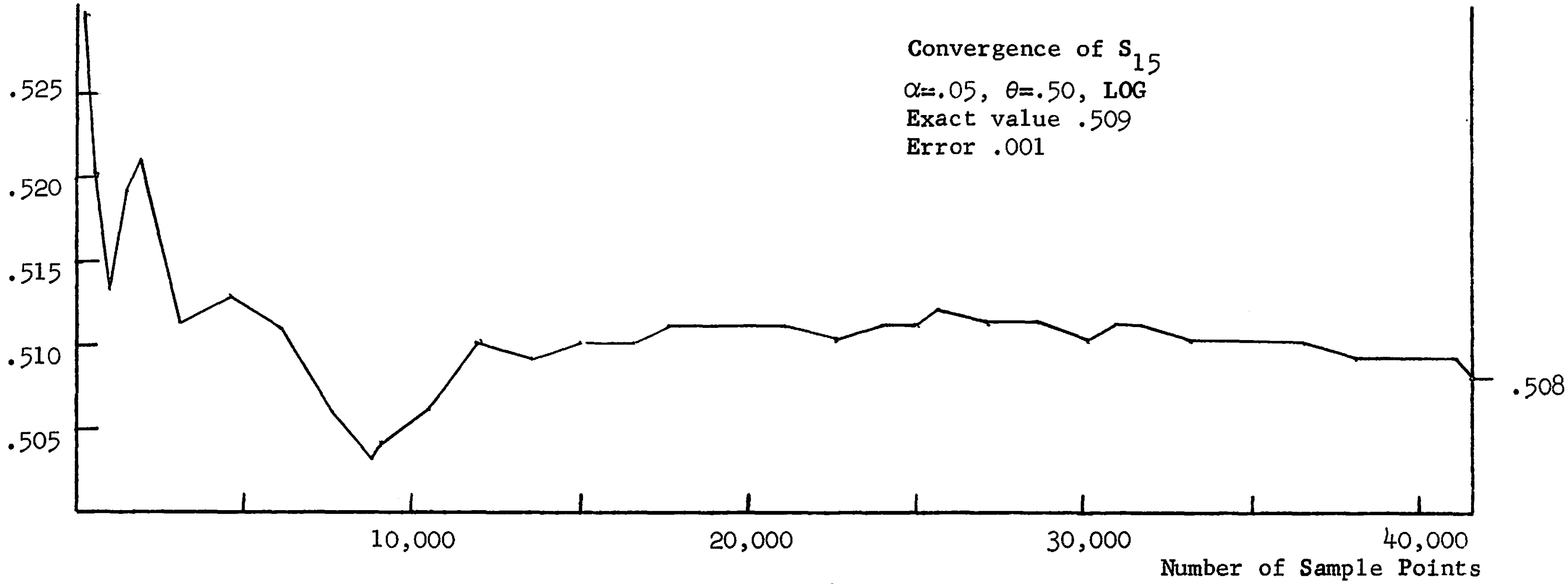


Figure 2.6

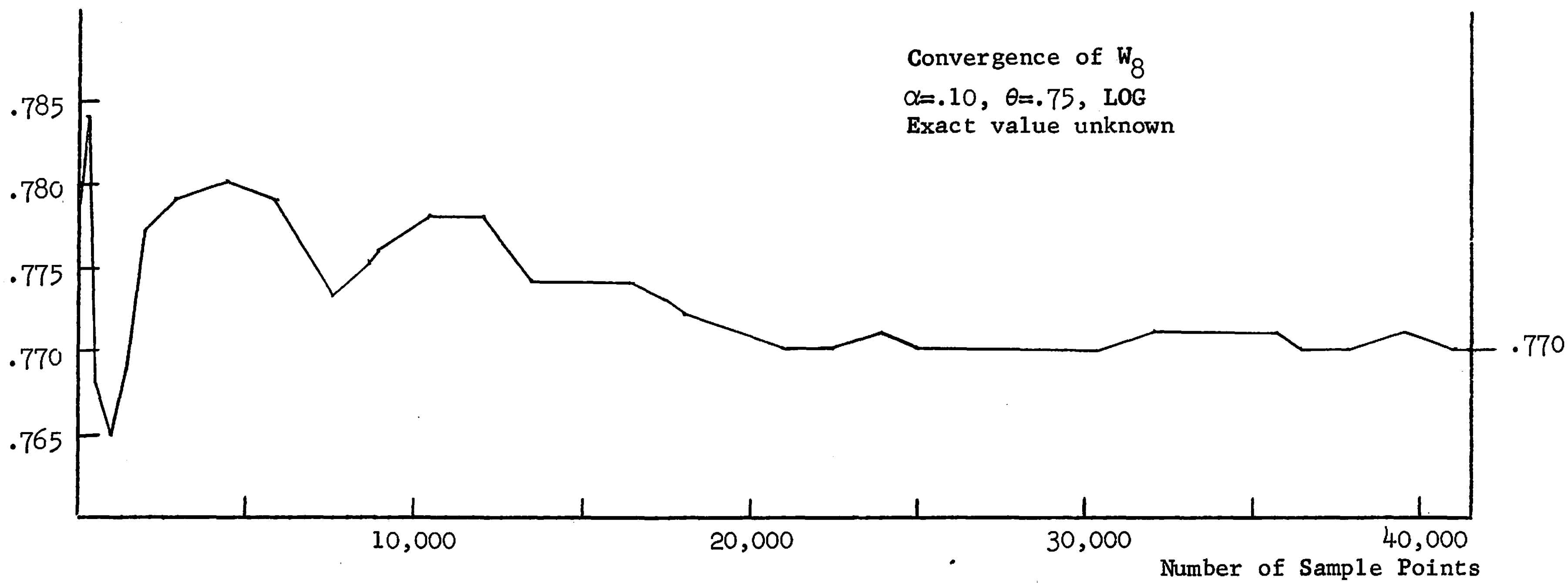
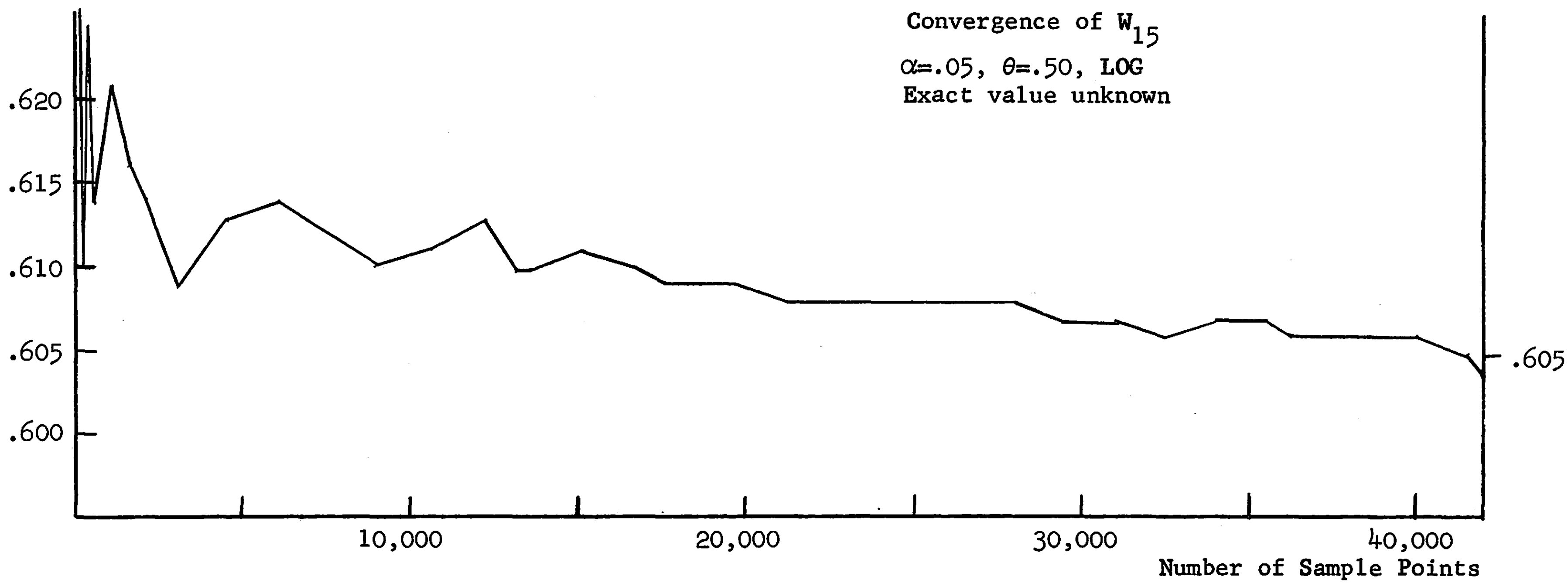


Figure 2.5



Convergence of W_{15}
α=.05, θ=.50, LOG
Exact value unknown
.620
.615
.610
.605
.600
.605
10,000
20,000
30,000
40,000
Number of Sample Points

Figure 2.7

some unknown integrals. These charts support what Table 2.1 indicates: the closer p is to $\frac{1}{2}$ the more sample points are needed for precise estimates. In most cases the estimates are fairly stable after 30,000 sample points. Figure 2.7 could be disconcerting until compared with figure 2.6; these two estimates are highly dependent.

2.4 Calculations

Generation of the Random Sample Points

Central to the procedure of computation was the necessity to generate random samples of size sixteen from U(0,1). First attempts to generate these samples was by means of a pseudo-random number generator [8]. In a trial run attempting to estimate the known null distribution of W_{15} unsatisfactory results were obtained. There appeared to be a serial correlation of some kind producing too many small values of the statistic. However, no attempt at formal analysis was made. Since at least one other user has also reported unsatisfactory results with this generator, it was decided to find some other source of numbers. This source was the table of one million random digits produced by the Rand Corporation. This table was produced from a rerandomized set of digits produced by an electronic "roulette wheel". The reader is referred to [10] for full details or to [13] for a review of [10]. The table was obtained on magnetic tape and used as input data. The tape actually used was a slight abbreviation of the table, with only about 889,000 digits. The tape consisted of 17,796 records, or card images, with 50 digits per record.

To help explain how the samples of sixteen numbers were assembled, some notation used in computer input/output coding will be introduced:

wX means skip w columns in the card image now being read, i.e., skip w digits on the present record, wF6.6 means read, from the present record, w numbers consisting of 6 decimal digits each with all digits to the right of the imposed decimal point,

/ means leave the present record and go to the next.

It should also be pointed out that an instruction to begin reading data will also initiate a new record.

The samples from U(0,1) were read in the following manner. It was decided that six digits per number would provide sufficiently low probability of a tie within a sample. Then for each of the following expressions the tape was started on a random record number less than 500 and exhausted:

```
(8F6.6/8F6.6)
(1X,8F6.6/1X,8F6.6)
(2X,8F6.6/2X,8F6.6)
(3X,7F6.6/3X,7F6.6/3X,2F6.6)
(4X,7F6.6/4X,7F6.6/4X,2F6.6)
(5X,7F6.6/5X,7F6.6/5X,2F6.6)
```

This procedure yielded 42,500 effectively uncorrelated samples of 16 independent observations from U(0,1) to six places of accuracy.

The Procedure of Calculation

Sixteen observations from U(0,1) provided one sample point for each of the 540 integrals estimated. Fifteen of the sixteen were transformed to samples from each of F_n, F_D, and F_L with $\theta = 0$. A shift parameter was then added to produce the desired alternative; the necessary rankings were then carried out. Next the statistics were calculated and the randomized decision procedure executed with the sixteenth U(0,1) observation. The appropriate counters were then increased and the entire procedure repeated by reading another sample of size sixteen from U(0,1). Flow charts are presented in the appendix.

Because of the necessity for equality tests in the randomized procedures, it was necessary to make sure that machine representations of computed statistics falling on the boundary value of a critical region coincided bit by bit with the machine representation of the stated value of the critical value. No problem is presented in this regard by W_n or S_n since all possible values are integral (and small). However, V_n can take on non-integral values and special care is necessary.

III SOME USES OF TABLE I

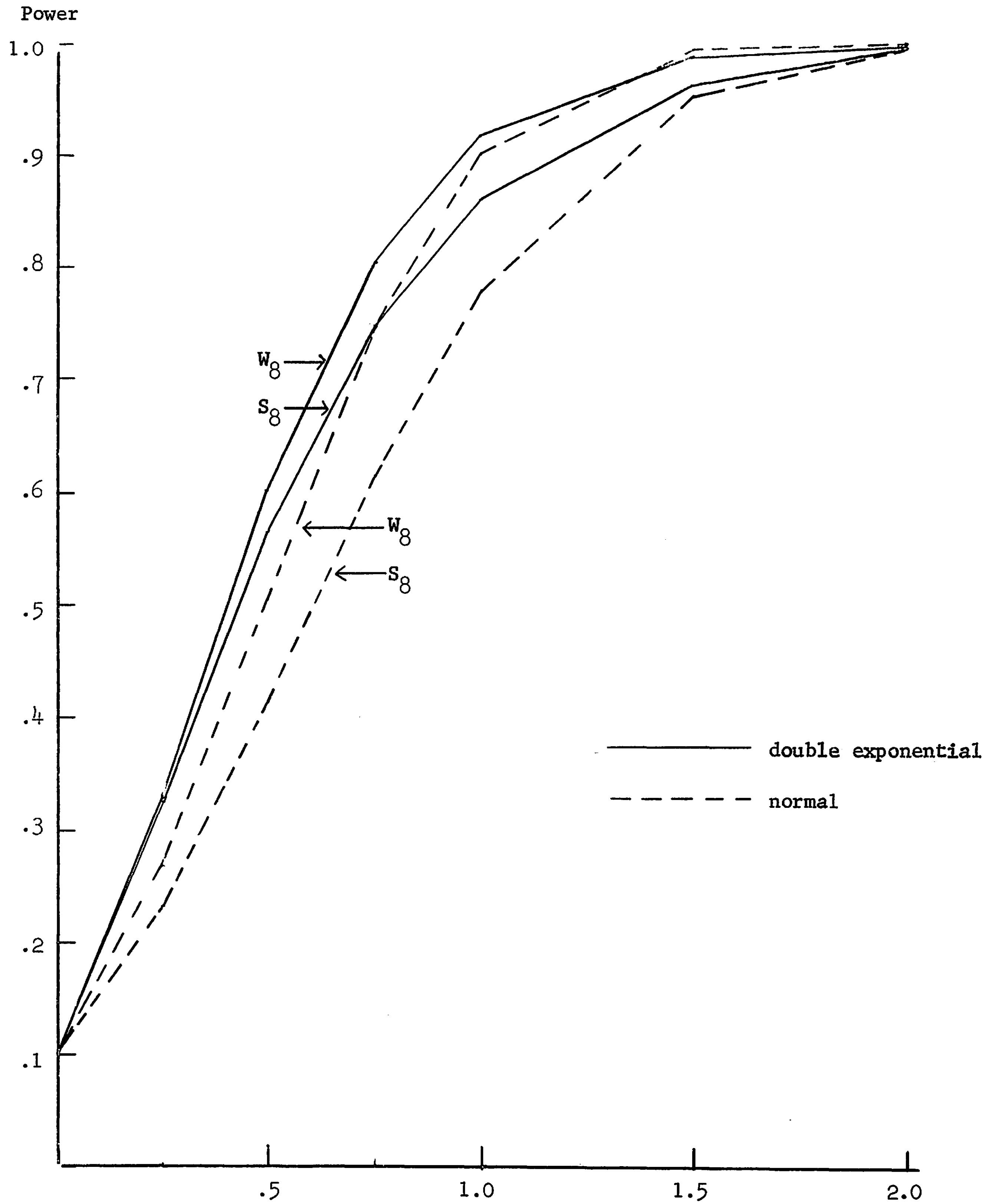
3.1 Introduction

Chapter III contains a very brief attempt to illustrate some of the information contained in Table I. All possible approaches certainly have not been exhausted; and the approaches taken here could be expanded. An evaluation is made of Hájek's large sample theory applied to small samples. There is also a comparison of rank tests on the basis of power. Comparison with parametric techniques, especially for non-normal alternatives, is one example of an interesting approach not taken here.

3.2 Evaluation of Some Large Sample Theory Applied to Small Samples

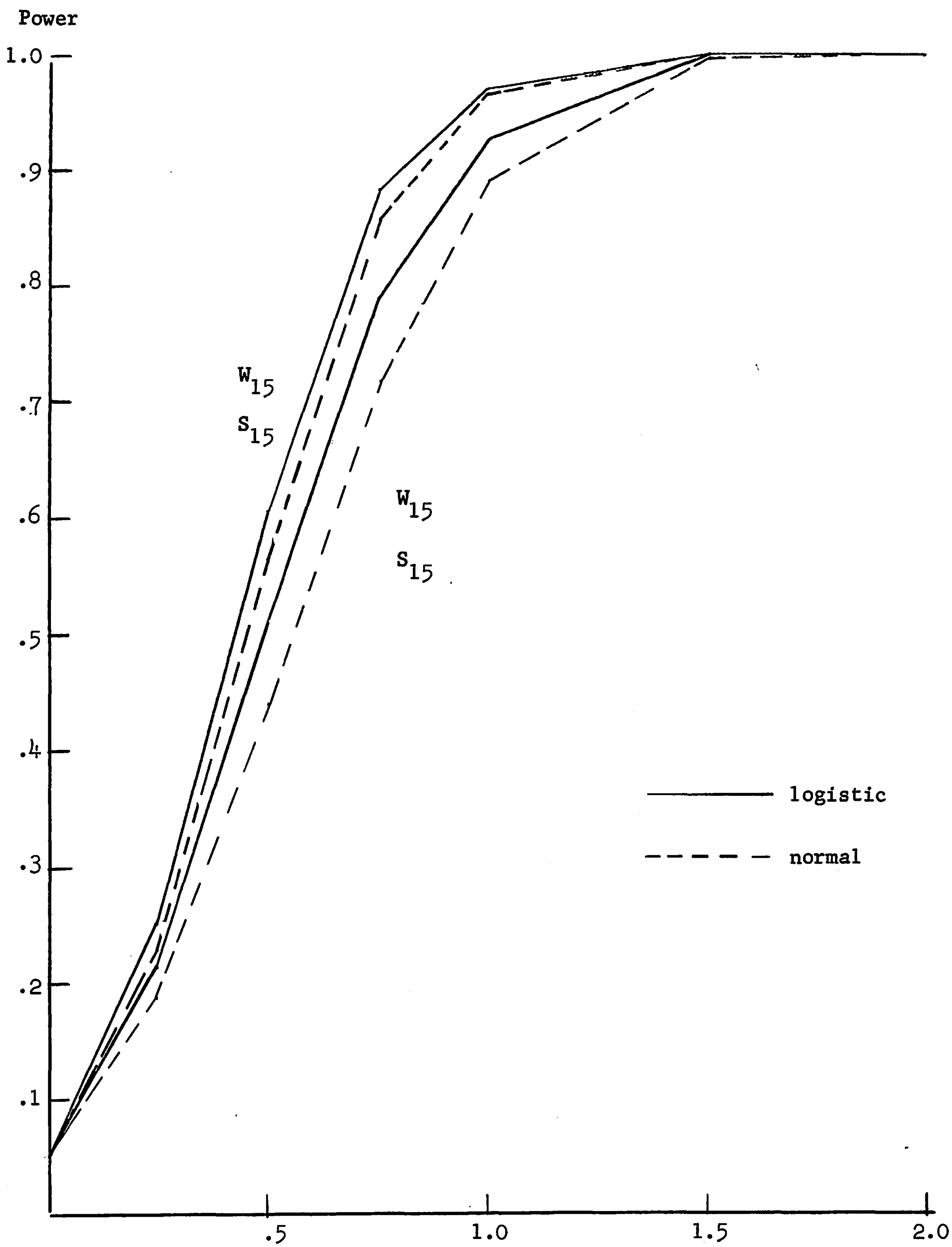
As mentioned in section 1.3, Hájek [4] has given a theory that specifies a test with certain optimum asymptotic properties for a specified translation family. The question can be raised as to how appropriate these tests are when applied to small samples. A partial answer can be had for this question by a brief glance at the tables, for example, on p. 41. It will be found that for $n = 8$ the test suggested by the large sample theory is not the best test. This theory suggests that S_8 should provide a best test for the double exponential translation family; however W_8 is a uniformly better test, with the difference in power running as high as .059. Also for $\theta \geqq .75$, V_8 is more powerful than S_8 for the double exponential. It should be noted, however, that for $\theta = .25$ and $n \geqq 10$ that S_n is most powerful for the double exponential. The case of $n = 8$ is partially illustrated in figure 3.1.

An interesting point to bring in here is that W_n is usually more powerful for double exponential alternatives than logistic alternatives.



Power of W_n and S_n: $n = 8$, $\alpha = .10$

Figure 3.1



Power of W_n and S_n: $n = 15$, $\alpha = .05$

Figure 3.2

The point of interest is that W_n is asymptotically locally most powerful, as well as apparently uniformly most powerful for small samples, for the logistic family. Likewise, V_n, at least for θ small, seems to be more powerful for double exponential and logistic alternatives than for normal alternatives. S_n is conspicuous by the fact that the distribution from which it is derived in Hájek's theory is the distribution for which it has the most power.

3.3 Choice of a Rank Test

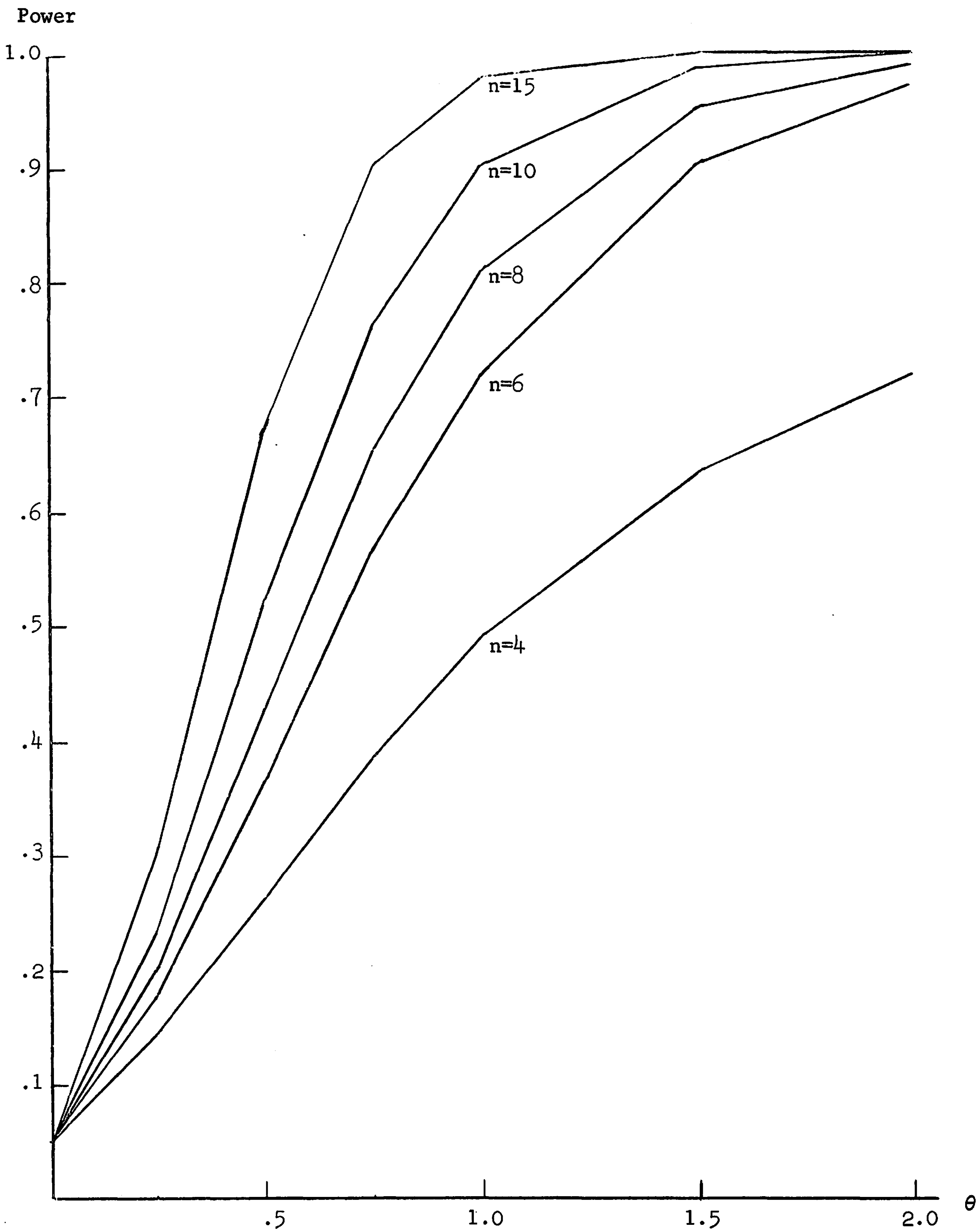
One conclusion that may be drawn from studying Table I is that the Wilcoxon test is the most desirable test to use when evaluated from the point of view of both power and ease of calculation. W_n is usually the most powerful or very close to the most powerful of the three tests considered. The major exception is for $n = 6$, $\alpha = .05$, $\theta = .25$ with the double exponential family; in examining these cases one finds a maximum difference in power of .008 in favor of the sign test. Other exceptions are some normal alternatives; among these there is a maximum difference in power of .009 in favor of the van der Waerden test. In the case of the van der Waerden versus the Wilcoxon, an experimenter may feel that the relative ease of computation of W_n makes up for the minor loss of power.

Comparison of the Wilcoxon test with the van der Waerden shows their power to be roughly the same. They are so close, for the most part, that it is impractical to try to display their difference graphically. However, the Wilcoxon, on the whole, seems to be considerably more powerful than the sign test with a maximum difference of .134 in favor of W_n. Two selected cases are illustrated in figures 3.2 and 3.3. Figure 3.1

suggests it may well be worthwhile to try to compare W_n and Students t for non-normal alternatives.

The maximum difference in power is .141 and occurs for V_n and S_n with $n = 10$, $\alpha = .10$, $\theta = .75$, normal alternative. The difference in V_n and W_n here is .007.

Figure 3.4 illustrates the power of W_n for double exponential alternatives with various sample sizes.



Power of the Wilcoxon Test for Double Exponential Alternatives: $\alpha = .05$

Figure 3.3

APPENDIX

Approximation of Critical Regions for V_{15}

One of the chief problems in tabulating the distribution or cirtical values of V_n is its very large number of possible distinct values. For $n = 8$, for example, there are 256 values, compared to 36 for W_8. At $n = 15$ there are 3200 possible distinct values for V_n; so the calculation of the critical regions for V_{n15} presented some problems. It was feared that the time required of either hand calculation or the analysis of programming required of computer calculation would extend the project beyond the required completion date; so a procedure for approximation was developed.

Since V_n is asymptotically normal, one might consider using a normal approximation for $n > 10$. However, the comparison with exact values for $n = 8$ and 10 showed a systematic deviation; in each case the normal approximation was too large. After studying van der Waerden's examination of the two sample analogue of V_n [14] it was concluded that the deviation here is due mainly to the relatively large terms

$$\Phi^{-1}\left(\tfrac{1}{2} + \frac{n-1}{2(n+1)}\right) \quad \text{and} \quad \Phi^{-1}\left(\tfrac{1}{2} + \frac{n}{2(n+1)}\right) \quad ,$$

which may or may not be included in V_n. An improved approximation can then be obtained by separating these terms from V_n.

Put

$$\text{Sign } X_k = \begin{cases} 0 \text{ if } X_k < 0 \\ 1 \text{ if } X_k \geq 0 \end{cases}$$

$$V_n^m = \sum_{k=1}^{m} \Phi^{-1}\left(\tfrac{1}{2} + \frac{k}{2(n+1)}\right) \text{Sign } X_k \ , \quad 1 \leq m \leq n$$

$$\psi_n(k) = \Phi^{-1}\left(\tfrac{1}{2} + \frac{k}{2(n+1)}\right)$$

$$m_1 = X_i \text{ where } |X_i| = \max(|X_1|, \ldots, |X_n|)$$

$$m_2 = X_i \text{ where } |X_i| = \max(\text{the sequence } |X_1|, \ldots, |X_n| \text{ with } a_1 \text{ deleted})$$

Then we have $V_n^n = V_n$, $V_n = \sum_{X_k>0} \psi_n(k)$

and
$$E(V_n^m) = \tfrac{1}{2}\sum_{k=1}^{m} \psi_n(k)$$

$$\mathrm{Var}(V_n^m) = \tfrac{1}{4}\sum_{k=1}^{m} \psi_n^2(k) \quad .$$

Note "Sign" refers to a function and "sign" to algebraic sign. The algebraic signs of m_1 and m_2 are independent and $p(m_i < 0) = p(m_i \geq 0) = \tfrac{1}{2}$, $i = 1,2$. So

(A1)
$$p(m_i \geq 0,\ m_2 \geq 0) = p(m_1 \geq 0, m_2 < 0) = p(m_1 < 0, m_2 \geq 0)$$
$$= p(m_1 < 0, m_2 < 0) = \tfrac{1}{4}.$$

Now the problem is that we want k in $P(V_n \leq k) = \alpha$ where P is not known exactly. $P(V_n \leq k)$ can be rewritten, conditioning on the signs of m_1 and m_2:

(A2)
$$P(V_n \leq k) = \tfrac{1}{4}P(V_n \leq k|m_1 \geq 0,\ m_2 \geq 0) + \tfrac{1}{4}P(V_n \leq k|m_1 \geq 0, m_2 < 0)$$
$$+ \tfrac{1}{4}P(V_n \leq k|m_1 < 0, m_2 \geq 0) + \tfrac{1}{4}P(V_n \leq k|m_1 < 0, m_2 < 0)$$
$$= \tfrac{1}{4}P[V_n^{n-2} \leq k-\psi_n(n)-\psi_n(n-1)] + \tfrac{1}{4}P[V_n^{n-2} \leq k-\psi_n(n)]$$
$$+ \tfrac{1}{4}P[V_n^{n-2} \leq k-\psi_n(n-1)] + \tfrac{1}{4}P[V_n^{n-2} \leq k] \quad .$$

Now standardizing the arguments of P in (A2) by $E(V_n^{n-2})$ and $\sqrt{\mathrm{Var}\ V_n^{n-2}}$ and replacing P by Φ we get

(A3)
$$\Phi\left(\frac{k-\psi_n(n)-\psi_n(n-1)-E(V_n^{n-2})}{\sqrt{\mathrm{Var}\ V_n^{n-2}}}\right) + \Phi\left(\frac{k-\psi_n(n)-E(V_n^{n-2})}{\sqrt{\mathrm{Var}\ V_n^{n-2}}}\right)$$
$$+ \Phi\left(\frac{k-\psi_n(n-1)-E(V_n^{n-2})}{\sqrt{\mathrm{Var}\ V_n^{n-2}}}\right) + \Phi\left(\frac{k-E(V_n^{n-2})}{\sqrt{\mathrm{Var}\ V_n^{n-2}}}\right) = 4\alpha \ .$$

Solving (A3) for k yields the desired approximation.

An alternative method of approximation is to condition on the sign of m_1 only, giving

(A4)
$$\Phi\left(\frac{k-\psi_n(n)-E(V_n^{n-1})}{\sqrt{\mathrm{Var}\ V_n^{n-1}}}\right) + \Phi\left(\frac{k-E(V_n^{n-1})}{\sqrt{\mathrm{Var}\ V_n^{n-1}}}\right) = 2\alpha.$$

This approximation does not seem to give as good results as (A3). Table A1 compares the approximations with the exact value for the case n = 10, α = .10.

TABLE A1

Method	Value	Diff with exact	Diff in pr mass with exact
Exact	5.6286		
A3	5.6328	.0042	.0014
A4	5.610	.019	.0039
Normal	5.5875	.0411	.0055

For n = 15, A3 was solved for k to 5 places of accuracy for α = .05 and .10. Table A2 shows the experimental significance levels obtained using these solutions for critical values; the table also shows some results for the straightforward normal approximation.

TABLE A2

Method	Number of sample points	Exp Sig Level for .05	.10
A3	42,500	.050	.099
A3	16,500	.050	.101
Normal	16,000	.052	.104

This approximation procedure was inspired by van der Waerden's procedure for the two sample case.

Flow Charts

Figure A1 presents a flow chart of the main computer program used in calculating the power functions. Figure A2 illustrates a subroutine referred to in figure A1.

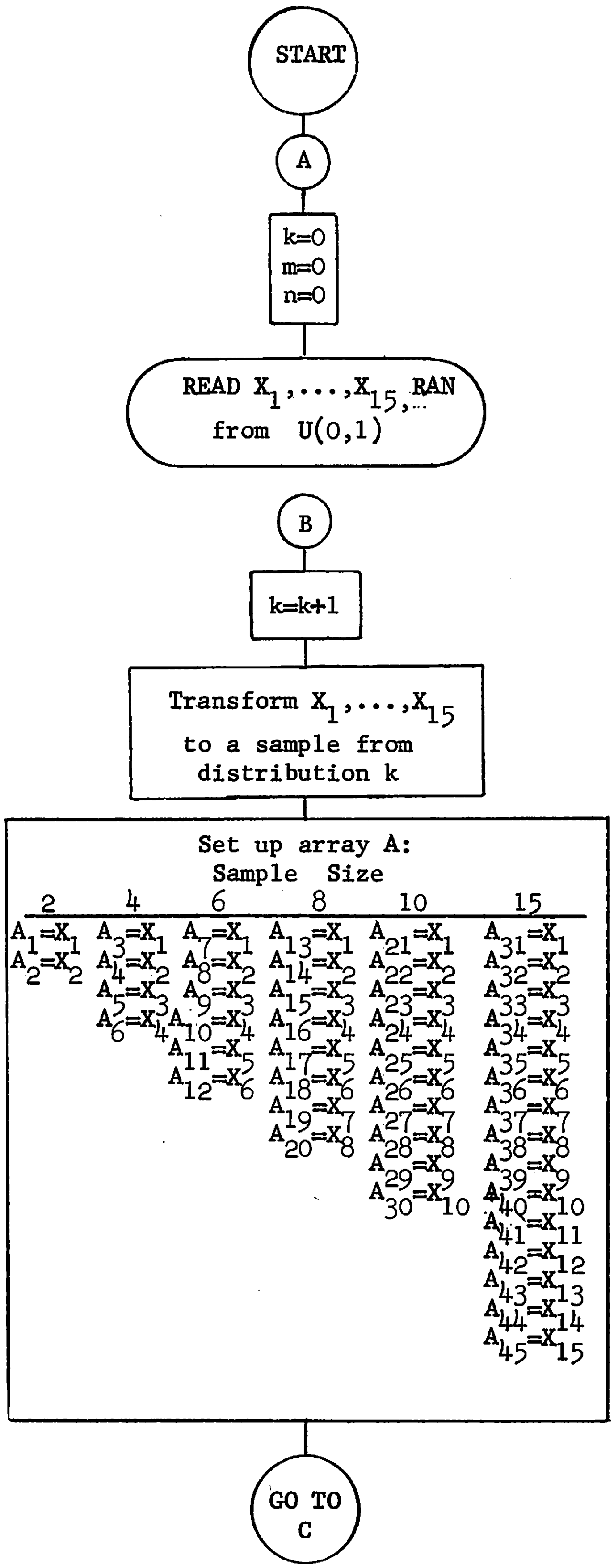

START
A
k=0
m=0
n=0
READ X1,...,X15, RAN
from U(0,1)
B
k=k+1
Transform X1,...,X15
to a sample from
distribution k
Set up array A:
Sample Size
2 4 6 8 10 15
A1=X1 A2=X2
A3=X1 A4=X2 A5=X3 A6=X4
A7=X1 A8=X2 A9=X3 A10=X4 A11=X5 A12=X6
A13=X1 A14=X2 A15=X3 A16=X4 A17=X5 A18=X6 A19=X7 A20=X8
A21=X1 A22=X2 A23=X3 A24=X4 A25=X5 A26=X6 A27=X7 A28=X8 A29=X9 A30=X10
A31=X1 A32=X2 A33=X3 A34=X4 A35=X5 A36=X6 A37=X7 A38=X8 A39=X9 A40=X10 A41=X11 A42=X12 A43=X13 A44=X14 A45=X15
GO TO
C

Figure A1(a)

* Arrays NSG and R are arranged conceptually like array A, i.e., with columns of varying lenght.

** The constants relating to W_n, S_n, and and V_n necessary for RANCT are explained on the following page.

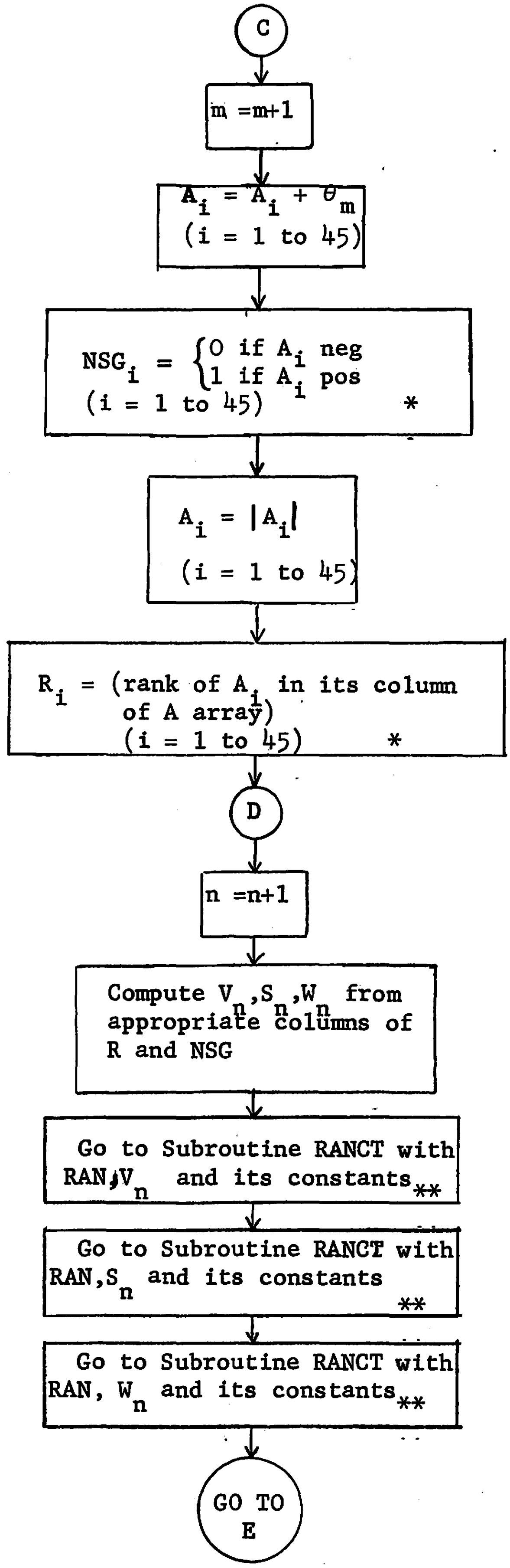


Figure A1(b)

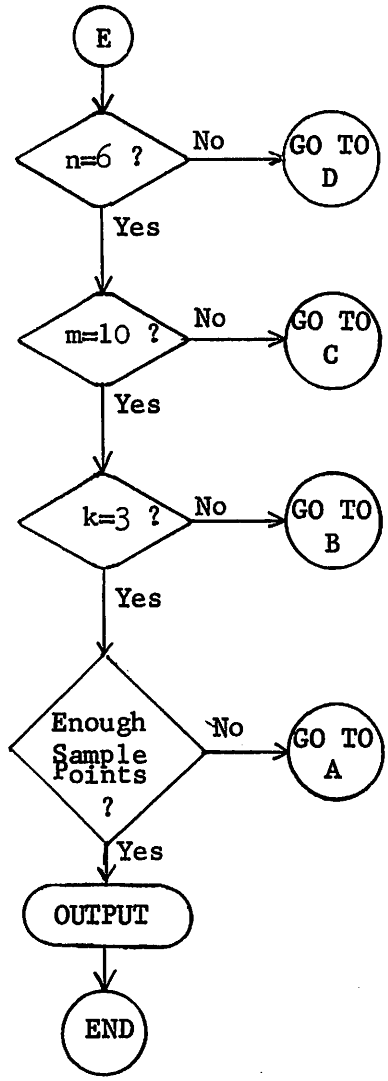


Figure A1(c)

The arguments of Subroutine RANCT are listed below. A flow chart of RANCT follows on the next page.

SUBROUTINE RANCT(ST,BV10,BV5,P10,P5,A10,A5,RAN)

ST = the value of the particular statistic in question

BV10 = the boundary point of the .10 critical region for ST

BV5 = the boundary point of the .05 critical region for ST

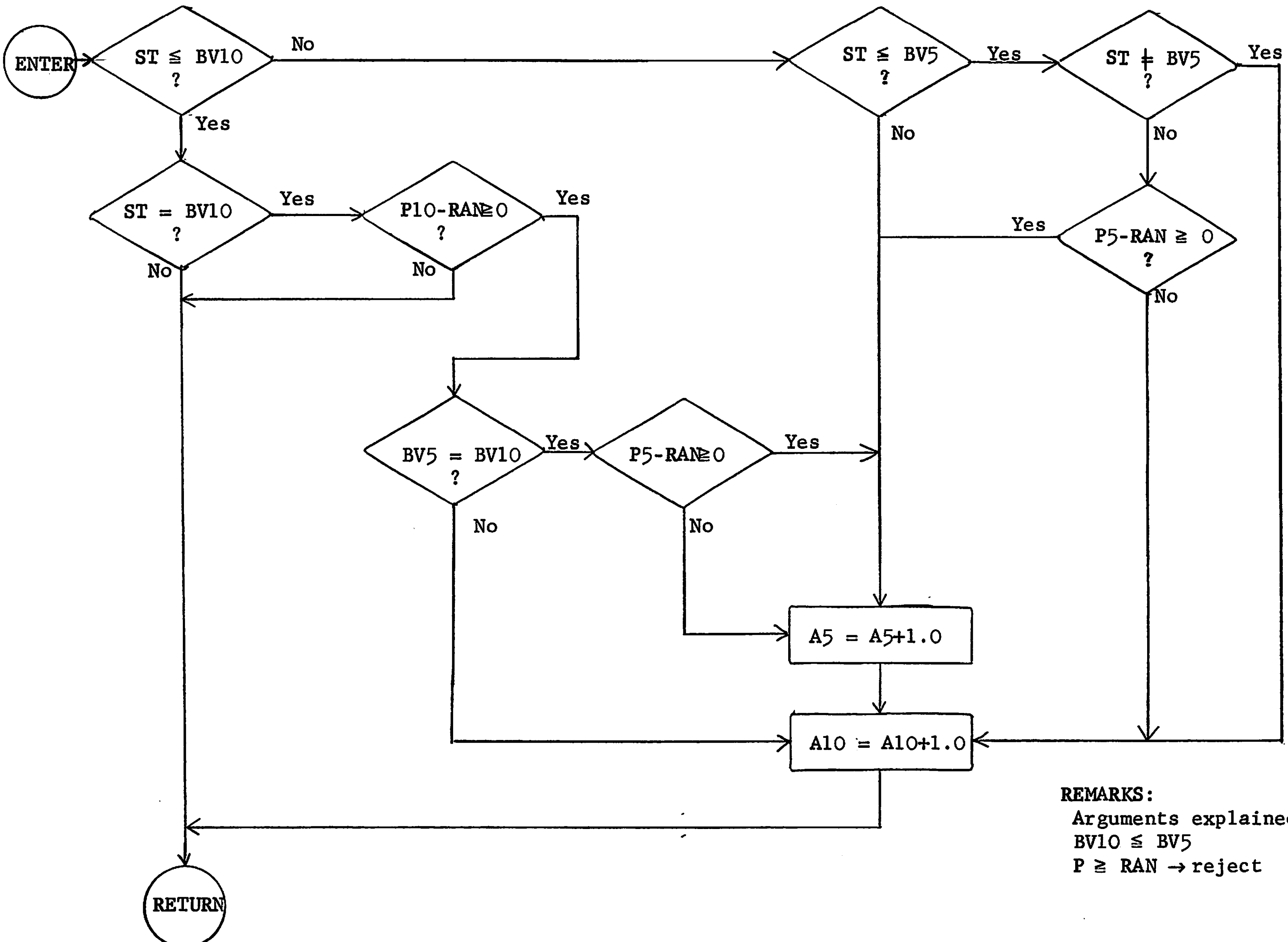
P10 = the probability of rejection at BV10

P5 = the probability of rejection at BV5

A10 = the counter increased by 1 if ST falls in .10 critical region

A5 = the counter increased by 1 if ST falls in .05 critical region

RAN = the 16^{th} U(0,1) observation; used for randomized decisions



REMARKS:
Arguments explained on p.36
BV10 ≦ BV5
P ≧ RAN → reject

SUBROUTINE RANCT(ST,BV10,BV5,P10,P5,A10,A5,RAN)

Figure A2

Table I

Sample Size = 2

		NORM	DE	LOG	NORM	DE	LOG
			θ = .00			θ = .25	
α = .05	VDW	.050	.050	.050	.070	.083	.073
	SIGN	.050	.050	.050	.070	.083	.073
	WIL	.050	.050	.050	.070	.083	.073
α = .10	VDW	.100	.100	.100	.144	.169	.149
	SIGN	.100	.100	.100	.144	.169	.149
	WIL	.100	.100	.100	.144	.169	.149
			θ = .50			θ = .75	
α = .05	VDW	.095	.112	.101	.118	.136	.126
	SIGN	.095	.112	.101	.118	.136	.126
	WIL	.095	.112	.101	.118	.136	.126
α = .10	VDW	.193	.228	.204	.241	.276	.255
	SIGN	.193	.228	.204	.241	.276	.255
	WIL	.193	.228	.204	.241	.276	.255
			θ = 1.00			θ = 1.50	
α = .05	VDW	.141	.154	.147	.173	.176	.175
	SIGN	.141	.154	.147	.173	.176	.175
	WIL	.141	.154	.147	.173	.176	.175
α = .10	VDW	.285	.311	.298	.350	.354	.353
	SIGN	.285	.311	.298	.350	.354	.353
	WIL	.285	.311	.298	.350	.354	.353
			θ = 2.00			θ = 2.50	
α = .05	VDW	.190	.188	.189	.197	.194	.196
	SIGN	.190	.188	.189	.197	.194	.196
	WIL	.190	.188	.189	.197	.194	.196
α = .10	VDW	.383	.378	.381	.397	.390	.394
	SIGN	.383	.378	.381	.397	.390	.394
	WIL	.383	.378	.381	.397	.390	.394
			θ = 3.00			θ = 4.00	
α = .05	VDW	.199	.197	.198	.199	.199	.199
	SIGN	.199	.197	.198	.199	.199	.199
	WIL	.199	.197	.198	.199	.199	.199
α = .10	VDW	.400	.396	.398	.401	.400	.401
	SIGN	.400	.396	.398	.401	.400	.401
	WIL	.400	.396	.398	.401	.400	.401

(42500)

Table I

Sample Size = 4

		NORM	DE	LOG	NORM	DE	LOG
			θ = .00			θ = .25	
α = .05	VDW	.050	.050	.050	.104	.144	.113
	SIGN	.050	.050	.050	.104	.144	.113
	WIL	.050	.050	.050	.105	.145	.114
α = .10	VDW	.100	.100	.100	.197	.249	.210
	SIGN	.099	.099	.099	.180	.235	.193
	WIL	.101	.101	.101	.198	.250	.211
			θ = .50			θ = .75	
α = .05	VDW	.186	.261	.208	.291	.380	.326
	SIGN	.186	.261	.208	.291	.380	.326
	WIL	.187	.263	.209	.293	.382	.329
α = .10	VDW	.327	.419	.358	.482	.579	.552
	SIGN	.290	.386	.319	.422	.529	.466
	WIL	.329	.421	.360	.485	.581	.524
			θ = 1.00			θ = 1.50	
α = .05	VDW	.406	.484	.444	.612	.629	.625
	SIGN	.406	.484	.444	.612	.629	.625
	WIL	.409	.488	.447	.617	.634	.629
α = .10	VDW	.637	.707	.671	.859	.865	.865
	SIGN	.561	.649	.604	.793	.812	.807
	WIL	.639	.708	.673	.861	.867	.867
			θ = 2.00			θ = 2.50	
α = .05	VDW	.734	.714	.724	.784	.759	.770
	SIGN	.734	.714	.724	.784	.759	.770
	WIL	.739	.719	.729	.790	.764	.776
α = .10	VDW	.958	.939	.949	.990	.973	.981
	SIGN	.926	.904	.916	.980	.953	.965
	WIL	.959	.940	.950	.990	.974	.981
			θ = 3.00			θ = 4.00	
α = .05	VDW	.799	.781	.790	.804	.798	.801
	SIGN	.799	.781	.790	.804	.798	.801
	WIL	.805	.787	.796	.809	.804	.807
α = .10	VDW	.998	.987	.993	1.000	.997	.999
	SIGN	.996	.976	.986	1.000	.994	.998
	WIL	.998	.988	.993	1.000	.997	.999

(42500)

Table I

Sample Size = 6

		NORM	DE	LOG	NORM	DE	LOG
			θ = .00			θ = .25	
	VDW						
α = .05	SIGN	.051	.051	.051	.115	.165	.126
	WIL	.051	.051	.051	.135	.177	.147
	VDW						
α = .10	SIGN	.101	.101	.101	.215	.294	.234
	WIL	.103	.103	.103	.234	.282	.248
			θ =.50			θ = .75	
	VDW						
α = .05	SIGN	.217	.317	.248	.355	.471	.400
	WIL	.274	.364	.306	.463	.562	.504
	VDW						
α = :10	SIGN	.372	.507	.416	.554	.682	.607
	WIL	.429	.504	.457	.642	.697	.671
			θ = 1.00			θ = 1.50	
	VDW						
α = .05	SIGN	.504	.602	.551	.765	.787	.781
	WIL	.658	.719	.689	.916	.904	.911
	VDW						
α = .10	SIGN	.716	.804	.761	.914	.927	.923
	WIL	.815	.829	.826	.975	.954	.965
			θ = 2.00			θ = 2.50	
	VDW						
α = .05	SIGN	.916	.891	.905	.977	.945	.960
	WIL	.989	.972	.981	.999	.992	.996
	VDW						
α = .10	SIGN	.980	.972	.977	.995	.988	.992
	WIL	.999	.989	.995	1.000	.997	.999
			θ = 3.00			θ = 4.00	
	VDW						
α = .05	SIGN	.995	.973	.984	1.000	.993	.997
	WIL	1.000	.998	.999	1.000	1.000	1.000
	VDW						
α = .10	SIGN	.999	.995	.997	1.000	.999	1.000
	WIL	1.000	.999	1.000	1.000	1.000	1.000

(42500)

Table I

Sample Size = 8

		NORM	DE	LOG	NORM	DE	LOG
			θ = .00			θ = .25	
α = .05	VDW	.050	.050	.050	.157	.196	.169
	SIGN	.052	.052	.052	.136	.205	.151
	WIL	.051	.051	.051	.158	.200	.170
α = .10	VDW	.101	.101	.101	.266	.310	.280
	SIGN	.102	.102	.102	.230	.322	.252
	WIL	.100	.100	.100	.264	.325	.281
			θ = .50			θ = .75	
α = .05	VDW	.346	.418	.376	.593	.639	.616
	SIGN	.280	.420	.322	.473	.624	.535
	WIL	.346	.428	.378	.592	.650	.620
α = .10	VDW	.507	.569	.531	.749	.768	.760
	SIGN	.412	.563	.461	.612	.745	.669
	WIL	.503	.600	.537	.745	.804	.771
			θ = 1.00			θ = 1.50	
α = .05	VDW	.799	.799	.803	.980	.948	.966
	SIGN	.665	.774	.720	.917	.931	.927
	WIL	.796	.809	.806	.980	.953	.968
α = .10	VDW	.905	.888	.900	.995	.978	.988
	SIGN	.778	.860	.820	.954	.962	.960
	WIL	.902	.916	.911	.995	.989	.992
			θ = 2.00			θ = 2.50	
α = .05	VDW	1.000	.988	.995	1.000	.997	.999
	SIGN	.989	.982	.986	.999	.995	.997
	WIL	.999	.990	.996	1.000	.998	.999
α = .10	VDW	1.000	.995	.998	1.000	.999	1.000
	SIGN	.994	.991	.993	.999	.998	.999
	WIL	1.000	.999	1.000	1.000	1.000	1.000
			θ = 3.00			θ = 4.00	
α = .05	VDW	1.000	.999	1.000	1.000	1.000	1.000
	SIGN	1.000	.999	1.000	1.000	1.000	1.000
	WIL	1.000	.999	1.000	1.000	1.000	1.000
α = .10	VDW	1.000	1.000	1.000	1.000	1.000	1.000
	SIGN	1.000	.999	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000

(42500)

Table I

Sample Size = 10

		NORM	DE	LOG	NORM	DE	LOG
			θ = .00			θ = .25	
	VDW	.050	.050	.050	.177	.218	.190
α = .05	SIGN	.051	.051	.051	.154	.242	.173
	WIL	.051	.051	.051	.178	.234	.194
	VDW	.102	.102	.102	.294	.346	.310
α = .10	SIGN	.102	.102	.102	.247	.357	.272
	WIL	.102	.102	.102	.290	.356	.309
			θ = .50			θ = .75	
	VDW	.413	.484	.440	.693	.722	.709
α = .05	SIGN	.337	.509	.390	.567	.727	.636
	WIL	.414	.519	.451	.692	.761	.723
	VDW	.581	.644	.606	.831	.849	.838
α = .10	SIGN	.464	.637	.521	.691	.823	.748
	WIL	.575	.667	.611	.824	.869	.846
			θ = 1.00			θ = 1.50	
	VDW	.888	.869	.882	.995	.975	.987
α = .05	SIGN	.769	.863	.819	.963	.971	.969
	WIL	.887	.902	.896	.995	.988	.992
	VDW	.955	.945	.951	.999	.995	.997
α =.10	SIGN	.855	.922	.891	.984	.988	.987
	WIL	.951	.957	.956	.999	.997	.998
			θ = 2.00			θ = 2.50	
	VDW	1.000	.995	.999	1.000	.999	1.000
α = .05	SIGN	.997	.994	.996	1.000	.999	1.000
	WIL	1.000	.999	1.000	1.000	1.000	1.000
	VDW	1.000	1.000	1.000	1.000	1.000	1.000
α = .10	SIGN	.999	.998	.999	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000
			θ = 3.00			θ =4.00	
	VDW	1.000	1.000	1.000	1.000	1.000	1.000
α = .05	SIGN	1.000	1.000	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000
	VDW	1.000	1.000	1.000	1.000	1.000	1.000
α = .10	SIGN	1.000	1.000	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000

(42500)

Table I

Sample Size = 15

		NORM	DE	LOG	NORM	DE	LOG
			θ = .00			θ = .25	
α = .05	VDW	.050	.050	.050	.231	.286	.246
	SIGN	.049	.049	.049	.187	.308	.213
	WIL	.050	.050	.050	.227	.303	.250
α = .10	VDW	.099	.099	.099	.361	.421	.377
	SIGN	.099	.099	.099	.295	.441	.329
	WIL	.099	.099	.099	.355	.438	.378
			θ = .50			θ = .75	
α = .05	VDW	.571	.637	.596	.860	.874	.868
	SIGN	.437	.648	.508	.715	.867	.785
	WIL	.563	.672	.604	.854	.901	.878
α = .10	VDW	.717	.771	.736	.934	.939	.937
	SIGN	.578	.771	.644	.823	.929	.873
	WIL	.708	.798	.743	.929	.953	.942
			θ = 1.00			θ = 1.50	
α = .05	VDW	.976	.966	.972	1.000	.998	1.000
	SIGN	.898	.957	.932	.996	.997	.997
	WIL	.974	.978	.977	1.000	.999	1.000
α = .10	VDW	.992	.988	.990	1.000	1.000	1.000
	SIGN	.949	.982	.968	.999	.999	.999
	WIL	.991	.992	.992	1.000	1.000	1.000
			θ = 2.00			θ = 2.50	
α = .05	VDW	1.000	1.000	1.000	1.000	1.000	1.000
	SIGN	1.000	1.000	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000
α = .10	VDW	1.000	1.000	1.000	1.000	1.000	1.000
	SIGN	1.000	1.000	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000
			θ = 3.00			θ = 4.00	
α = .05	VDW	1.000	1.000	1.000	1.000	1.000	1.000
	SIGN	1.000	1.000	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000
α = .10	VDW	1.000	1.000	1.000	1.000	1.000	1.000
	SIGN	1.000	1.000	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000

(42500)

TABLE II

Table II

Sample Size = 2

		NORM	DE	LOG	NORM	DE	LOG
		θ = .00			θ = .25		
α = .05	VDW	.049	.049	.049	.070	.083	.073
	SIGN	.049	.049	.049	.070	.083	.073
	WIL	.049	.049	.409	.070	.083	.073
α = .10	VDW	.100	.100	.100	.144	.169	.149
	SIGN	.100	.100	.100	.144	.169	.149
	WIL	.100	.100	.100	.144	.169	.149
		θ = .50			θ = .75		
α = .05	VDW	.095	.113	.101	.119	.136	.126
	SIGN	.095	.113	.101	.119	.136	.126
	WIL	.095	.113	.101	.119	.136	.126
α = .10	VDW	.193	.229	.205	.241	.276	.256
	SIGN	.193	.229	.205	.241	.276	.256
	WIL	.193	.229	.205	.241	.276	.256
		θ = 1.00			θ = 1.50		
α = .05	VDW	.141	.154	.147	.173	.176	.175
	SIGN	.141	.154	.147	.173	.176	.175
	WIL	.141	.154	.147	.173	.176	.175
α = .10	VDW	.285	.311	.298	.350	.355	.353
	SIGN	.285	.311	.298	.350	.355	.353
	WIL	.285	.311	.298	.350	.355	.353
		θ = 2.00			θ = 2.50		
α = .05	VDW	.190	.188	.189	.197	.194	.195
	SIGN	.190	.188	.189	.197	.194	.195
	WIL	.190	.188	.189	.197	.194	.195
α = .10	VDW	.384	.378	.381	.397	.391	.394
	SIGN	.384	.378	.381	.397	.391	.394
	WIL	.384	.378	.381	.397	.391	.394
		θ = 3.00			θ = 4.00		
α = .05	VDW	.199	.197	.198	.199	.199	.199
	SIGN	.199	.197	.198	.199	.199	.199
	WIL	.199	.197	.198	.199	.199	.199
α = .10	VDW	.401	.396	.398	.402	.400	.401
	SIGN	.401	.396	.398	.402	.400	.401
	WIL	.401	.396	.398	.402	.400	.401

(42000)

Table II

Sample Size = 4

		NORM	DE	LOG	NORM	DE	LOG
			θ = .00			θ = .25	
α = .05	VDW	.050	.050	.050	.104	.144	.113
	SIGN	.050	.050	.050	.104	.144	.113
	WIL	.050	.050	.050	.105	.145	.114
α = .10	VDW	.100	.100	.100	.198	.249	.210
	SIGN	.099	.099	.099	.180	.235	.193
	WIL	.101	.101	.101	.199	.250	.212
			θ = .50			θ = .75	
α = .05	VDW	.186	.261	.208	.291	.380	.327
	SIGN	.186	.261	.208	.291	.380	.327
	WIL	.187	.263	.210	.293	.382	.329
α = .10	VDW	.328	.419	.358	.483	.579	.522
	SIGN	.290	.386	.319	.423	.529	.466
	WIL	.330	.421	.360	.485	.581	.524
			θ = 1.00			θ = 1.50	
α = .05	VDW	.407	.484	.444	.612	.629	.625
	SIGN	.407	.484	.444	.612	.629	.625
	WIL	.409	.488	.447	.617	.634	.629
α = .10	VDW	.637	.707	.671	.860	.866	.865
	SIGN	.561	.649	.604	.793	.812	.807
	WIL	.640	.709	.673	.861	.867	.867
			θ = 2.00			θ = 2.50	
α = .05	VDW	.734	.714	.724	.784	.758	.770
	SIGN	.734	.714	.724	.784	.758	.770
	WIL	.739	.719	.729	.790	.764	.776
α = .10	VDW	.958	.939	.949	.990	.973	.981
	SIGN	.926	.904	.916	.980	.953	.965
	WIL	.959	.940	.950	.990	.974	.981
			θ = 3.00			θ = 4.00	
α = .05	VDW	.799	.781	.790	.803	.798	.801
	SIGN	.799	.781	.790	.803	.798	.801
	WIL	.805	.787	.796	.809	.804	.807
α = .10	VDW	.998	.987	.993	1.000	.997	.999
	SIGN	.996	.976	.986	1.000	.994	.998
	WIL	.998	.988	.993	1.000	.997	.999

(42000)

Table II

Sample Size = 6

		NORM	DE	LOG	NORM	DE	LOG
			θ = .00			θ = .25	
	VDW						
α = .05	SIGN	.051	.051	.051	.115	.166	.127
	WIL	.051	.051	.051	.135	.177	.147
	VDW						
α = .10	SIGN	.101	.101	.101	.215	.294	.234
	WIL	.103	.103	.103	.234	.282	.248
			θ = .50			θ = .75	
	VDW						
α = .05	SIGN	.218	.318	.248	.355	.471	.401
	WIL	.275	.365	.306	.463	.562	.504
	VDW						
α = .10	SIGN	.373	.507	.416	.555	.682	.608
	WIL	.429	.504	.457	.642	.697	.671
			θ = 1.00			θ = 1.50	
	VDW						
α = .05	SIGN	.505	.602	.551	.765	.787	.781
	WIL	.658	.719	.689	.916	.904	.911
	VDW						
α = .10	SIGN	.717	.804	.761	.914	.927	.924
	WIL	.815	.829	.826	.975	.954	.965
			θ = 2.00			θ = 2.50	
	VDW						
α = .05	SIGN	.916	.891	.905	.977	.945	.960
	WIL	.989	.972	.981	.999	.992	.996
	VDW						
α = .10	SIGN	.980	.972	.977	.995	.988	.992
	WIL	.999	.989	.995	1.000	.997	.999
			θ = 3.00			θ = 4.00	
	VDW						
α = .05	SIGN	.995	.973	.984	1.000	.993	.997
	WIL	1.000	.998	.999	1.000	1.000	1.000
	VDW						
α = .10	SIGN	.999	.995	.997	1.000	.999	1.000
	WIL	1.000	.999	1.000	1.000	1.000	1.000

(42000)

Table II

Sample Size = 8

		NORM	DE	LOG	NORM	DE	LOG
		θ = .00			θ = .25		
α = .05	VDW	.050	.050	.050	.157	.196	.169
	SIGN	.052	.052	.052	.136	.205	.151
	WIL	.051	.051	.051	.158	.200	.170
α = .10	VDW	.101	.101	.101	.266	.310	.280
	SIGN	.102	.102	.102	.230	.322	.252
	WIL	.101	.101	.101	.264	.325	.281
		θ = .50			θ = .75		
α = .05	VDW	.346	.418	.376	.593	.639	.617
	SIGN	.280	.421	.322	.473	.624	.535
	WIL	.346	.428	.378	.592	.650	.620
α = .10	VDW	.508	.569	.531	.750	.769	.761
	SIGN	.412	.563	.461	.613	.745	.669
	WIL	.504	.600	.538	.745	.804	.771
		θ = 1.00			θ = 1.50		
α = .05	VDW	.799	.799	.803	.980	.948	.966
	SIGN	.665	.774	.720	.917	.931	.927
	WIL	.767	.809	.806	.980	.953	.969
α = .10	VDW	.905	.888	.900	.995	.978	.988
	SIGN	.778	.860	.820	.954	.962	.960
	WIL	.902	.917	.911	.995	.989	.993
		θ = 2.00			θ = 2.50		
α = .05	VDW	1,000	.988	.995	1.000	.997	.999
	SIGN	.989	.982	.986	.999	.995	.998
	WIL	.999	.990	.996	1.000	.998	.999
α = .10	VDW	1.000	.995	.998	1.000	.999	1.000
	SIGN	.994	.991	.993	.999	.998	.999
	WIL	1.000	.999	1.000	1.000	1.000	1.000
		θ = 3.00			θ = 4.00		
α = .05	VDW	1.000	.999	1.000	1.000	1.000	1.000
	SIGN	1.000	.999	1.000	1.000	1.000	1.000
	WIL	1.000	.999	1.000	1.000	1.000	1.000
α = .10	VDW	1.000	1.000	1.000	1.000	1.000	1.000
	SIGN	1.000	.999	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000

(42000)

Table II

Sample Size = 10

		NORM	DE	LOG	NORM	DE	LOG
			θ = .00			θ = .25	
	VDW	.050	.050	.050	.178	.218	.190
α = .05	SIGN	.051	.051	.051	.154	.242	.173
	WIL	.051	.051	.051	.179	.234	.194
	VDW	.102	.102	.102	.294	.346	.310
α = .10	SIGN	.102	.102	.102	.247	.358	.272
	WIL	.102	.102	.102	.291	.357	.310
			θ = .50			θ = .75	
	VDW	.413	.484	.441	.694	.722	.709
α = .05	SIGN	.338	.509	.391	.568	.727	.636
	WIL	.414	.519	.451	.692	.762	.723
	VDW	.581	.645	.606	.831	.849	.838
α = .10	SIGN	.465	.637	.521	.691	.823	.749
	WIL	.575	.667	.611	.824	.869	.846
			θ = 1.00			θ = 1.50	
	VDW	.888	.869	.882	.995	.975	.987
α = .05	SIGN	.769	.863	.819	.963	.971	.969
	WIL	.887	.902	.896	.995	.988	.992
	VDW	.955	.946	.951	.999	.995	.997
α = .10	SIGN	.855	.922	.891	.984	.988	.987
	WIL	.951	.957	.956	.999	.997	.998
			θ = 2.00			θ = 2.50	
	VDW	1.000	.995	.999	1.000	.999	1.000
α = .05	SIGN	.997	.994	.996	1.000	.999	1.000
	WIL	1.000	.999	1.000	1.000	1.000	1.000
	VDW	1.000	1.000	1.000	1.000	1.000	1.000
α = .10	SIGN	.999	.998	.999	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000
			θ = 3.00			θ = 4.00	
	VDW	1.000	1.000	1.000	1.000	1.000	1.000
α = .05	SIGN	1.000	1.000	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000
	VDW	1.000	1.000	1.000	1.000	1.000	1.000
α = .10	SIGN	1.000	1.000	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000

(42000)

Table II

Sample Size = 15

		NORM	DE	LOG	NORM	DE	LOG
			θ = .00			θ = .25	
α = .05	VDW	.050	.050	.050	.231	.286	.246
	SIGN	.049	.049	.049	.187	.308	.213
	WIL	.050	.050	.050	.227	.303	.250
α = .10	VDW	.099	.099	.099	.361	.421	.377
	SIGN	.099	.099	.099	.295	.442	.329
	WIL	.099	.099	.099	.355	.439	.378
			θ = .50			θ = .75	
α = .05	VDW	.571	.637	.596	.860	.875	.868
	SIGN	.438	.649	.509	.715	.867	.785
	WIL	.563	.672	.605	.854	.901	.878
α = .10	VDW	.717	.772	.737	.934	.939	.937
	SIGN	.579	.772	.645	.823	.930	.874
	WIL	.709	.798	.743	.929	.954	.942
			θ = 1.00			θ = 1.50	
α = .05	VDW	.976	.966	.972	1.000	.998	1.000
	SIGN	.898	.957	.932	.996	.997	.997
	WIL	.974	.978	.977	1.000	.999	1.000
α = .10	VDW	.992	.988	.990	1.000	1.000	1.000
	SIGN	.949	.982	.968	.999	.999	.999
	WIL	.991	.992	.992	1.000	1.000	1.000
			θ = 2.00			θ = 2.50	
α = .05	VDW	1.000	1.000	1.000	1.000	1.000	1.000
	SIGN	1.000	1.000	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000
α = .10	VDW	1.000	1.000	1.000	1.000	1.000	1.000
	SIGN	1.000	1.000	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000
			θ = 3.00			θ = 4.00	
α = .05	VDW	1.000	1.000	1.000	1.000	1.000	1.000
	SIGN	1.000	1.000	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000
α = .10	VDW	1.000	1.000	1.000	1.000	1.000	1.000
	SIGN	1.000	1.000	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000

(42000)

TABLE III

Table III

Sample Size = 2

		NORM	DE	LOG	NORM	DE	LOG
			θ = .00			θ = .25	
α = .05	VDW	.060	.060	.060	.68	.070	.068
	SIGN	.060	.060	.060	.068	.070	.068
	WIL	.060	.060	.060	.068	.070	.068
α = .10	VDW	.094	.094	.094	.124	.138	.130
	SIGN	.094	.094	.094	.124	.138	.130
	WIL	.094	.094	.094	.124	.138	.130
			θ = .50			θ = .75	
α = .05	VDW	.086	.102	.088	.114	.146	.130
	SIGN	.086	.102	.088	.114	.146	.130
	WIL	.086	.102	.088	.114	.146	.130
α = .10	VDW	.170	.202	.184	.214	.258	.238
	SIGN	.170	.202	.184	.214	.258	.238
	WIL	.170	.202	.184	.214	.258	.238
			θ = 1.00			θ = 1.50	
α = .05	VDW	.148	.160	.154	.178	.180	.180
	SIGN	.148	.160	.154	.178	.180	.180
	WIL	.148	.160	.154	.178	.180	.180
α = .10	VDW	.264	.288	.274	.322	.324	.324
	SIGN	.264	.288	.274	.322	.324	.324
	WIL	.264	.288	.274	.322	.324	.324
			θ = 2.00			θ = 2.50	
α = .05	VDW	.204	.200	.202	.204	.204	.204
	SIGN	.204	.200	.202	.204	.204	.204
	WIL	.204	.200	.202	.204	.204	.204
α = .10	VDW	.364	.360	.362	.372	.368	.368
	SIGN	.364	.360	.362	.372	.368	.368
	WIL	.364	.360	.362	.372	.368	.368
			θ = 3.00			θ = 4.00	
α = .05	VDW	.204	.204	.204	.208	.204	.206
	SIGN	.204	.204	.204	.208	.204	.206
	WIL	.204	.204	.204	.208	.204	.206
α = .10	VDW	.378	.372	.374	.382	.378	.380
	SIGN	.378	.372	.374	.382	.378	.380
	WIL	.378	.372	.374	.382	.378	.380

(500)

Table III

Sample Size = 4

		NORM	DE	LOG	NORM	DE	LOG
			θ = .00			θ = .25	
α = .05	VDW	.040	.040	.040	.078	.116	.098
	SIGN	.040	.040	.040	.078	.116	.098
	WIL	.040	.040	.040	.078	.116	.098
α = .10	VDW	.068	.068	.068	.152	.208	.172
	SIGN	.078	.078	.078	.152	.204	.176
	WIL	.068	.068	.068	.152	.210	.172
			θ = .50			θ = .75	
α = .05	VDW	.152	.226	.180	.262	.364	.304
	SIGN	.152	.226	.180	.262	.364	.304
	WIL	.152	.228	.182	.264	.368	.308
α = .10	VDW	.282	.382	.314	.454	.562	.502
	SIGN	.256	.360	.290	.404	.504	.454
	WIL	.284	.384	.316	.454	.564	.502
			θ = 1.00			θ = 1.50	
α = .05	VDW	.392	.484	.428	.604	.620	.616
	SIGN	.392	.484	.428	.604	.620	.616
	WIL	.396	.488	.432	.608	.624	.620
α = .10	VDW	.620	.708	.676	.850	.854	.854
	SIGN	.540	.652	.588	.786	.800	.796
	SIL	.622	.708	.676	.850	.854	.854
			θ = 2.00			θ = 2.50	
α = .05	VDW	.742	.718	.724	.794	.772	.778
	SIGN	.742	.718	.724	.794	.772	.778
	WIL	.746	.722	.728	.798	.776	.782
α = .10	VDW	.956	.938	.948	.992	.978	.982
	SIGN	.920	.896	.900	.980	.952	.960
	WIL	.956	.938	.948	.992	.978	.982
			θ = 3.00			θ = 4.00	
α = .05	VDW	.812	.792	.800	.820	.812	.816
	SIGN	.812	.792	.800	.820	.812	.816
	WIL	.816	.796	.804	.824	.816	.820
α = .10	VDW	.996	.988	.992	1.000	.996	.998
	SIGN	.996	.976	.984	1.000	.996	.998
	WIL	.996	.988	.992	1.000	.996	.998

(500)

Table III

Sample Size = 6

		NORM	DE	LOG	NORM	DE	LOG
			θ = .00			θ = .25	
	VDW	.042	.042	.042	.132	.168	.140
α = .05	SIGN	.036	.036	.036	.096	.138	.120
	WIL	.042	.042	.042	.120	.160	.132
		.084	.084	.084	.232	.274	.250
α = .10	SIGN	.084	.084	.084	.208	.272	.228
	WIL	.808	.080	.080	.228	.264	.246
			θ = .50			θ = .75	
	VDW	.260	.358	.298	.460	.578	.520
α = .05	SIGN	.182	.288	.204	.322	.462	.376
	WIL	.248	.352	.284	.452	.570	.508
	VDW	.418	.510	.454	.624	.684	.660
α = .10	SIGN	.326	.462	.372	.512	.640	.560
	WIL	.404	.514	.452	.616	.678	.656
			θ = 1.00			θ = 1.50	
	VDW	.666	.714	.692	.904	.886	.894
α = .05	SIGN	.494	.620	.548	.770	.786	.782
	WIL	.658	.714	.692	.898	.882	.892
	VDW	.800	.818	.802	.970	.946	.956
α = .10	SIGN	.682	.798	.736	.902	.916	.912
	WIL	.792	.810	.798	.972	.948	.960
			θ = 2.00			θ = 2.50	
	VDW	.988	.964	.982	1.000	.996	.996
α = .05	SIGN	.912	.892	.898	.976	.938	.954
	WIL	.988	.962	.980	1.000	.996	.996
	VDW	1.000	.992	.996	1.000	.996	1.000
α = .10	SIGN	.982	.968	.974	.996	.990	.992
	WIL	1.000	.992	.996	1.000	.996	1.000
			θ = 3.00			θ = 4.00	
	VDW	1.000	1.000	1.000	1.000	1.000	1.000
α = .05	SIGN	.996	.972	.988	1.000	.994	.998
	WIL	1.000	1.000	1.000	1.000	1.000	1.000
	VDW	1.000	1.000	1.000	1.000	1.000	1.000
α = .10	SIGN	1.000	.994	.998	1.000	.998	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000

(500)

Table III

Sample Size = 8

		NORM	DE	LOG	NORM	DE	LOG
		θ = .00			θ = .25		
α = .05	VDW	.052	.052	.052	.148	.186	.158
	SIGN	.052	.052	.052	.130	.188	.148
	WIL	.052	.052	.052	.150	.198	.166
α = .10	VDW	.100	.100	.100	.256	.312	.282
	SIGN	.102	.102	.102	.228	.302	.252
	WIL	.092	.092	.092	.254	.324	.284
		θ = .50			θ = .75		
α = .05	VDW	.336	.426	.384	.582	.634	.612
	SIGN	.244	.404	.288	.456	.600	.524
	WIL	.330	.428	.378	.586	.638	.614
α = .10	VDW	.488	.566	.520	.716	.740	.740
	SIGN	.368	.536	.416	.592	.748	.674
	WIL	.488	.580	.522	.706	.774	.742
		θ = 1.00			θ = 1.50		
α = .05	VDW	.780	.790	.792	.970	.932	.946
	SIGN	.642	.766	.702	.888	.906	.902
	WIL	.766	.804	.782	.976	.940	.954
α = .10	VDW	.876	.868	.870	.992	.962	.978
	SIGN	.784	.864	.822	.942	.954	.950
	WIL	.878	.896	.880	.990	.978	.988
		θ = 2.00			θ = 2.50		
α = .05	VDW	1.000	.986	.996	1.000	.996	1.000
	SIGN	.982	.974	.978	1.000	.994	.994
	WIL	1.000	.990	.998	1.000	1.000	1.000
α = .10	VDW	1.000	.996	1.000	1.000	1.000	1.000
	SIGN	.988	.986	.986	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000
		θ = 3.00			θ = 4.00		
α = .05	VDW	1.000	1.000	1.000	1.000	1.000	1.000
	SIGN	1.000	.998	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000
α = .10	VDW	1.000	1.000	1.000	1.000	1.000	1.000
	SIGN	1.000	1.000	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000

(500)

Table III

Sample Size = 10

		NORM	DE	LOG	NORM	DE	LOG
		θ = .00			θ = .25		
α = .05	VDW	.050	.050	.050	.170	.206	.188
	SIGN	.052	.052	.052	.148	.216	.168
	WIL	.046	.046	.046	.164	.204	.180
α = .10	VDW	.108	.108	.108	.270	.328	.290
	SIGN	.106	.106	.106	.228	.318	.258
	WIL	.104	.104	.104	.254	.328	.272
		θ = .50			θ = .75		
α = .05	VDW	.406	.470	.422	.672	.688	.684
	SIGN	.280	.462	.332	.520	.698	.614
	WIL	.394	.510	.434	.670	.722	.690
α = .10	VDW	.562	.620	.592	.788	.810	.796
	SIGN	.410	.572	.462	.650	.800	.736
	WIL	.554	.658	.592	.786	.842	.800
		θ = 1.00			θ = 1.50		
α = .05	VDW	.854	.856	.872	.998	.968	.992
	SIGN	.756	.842	.800	.954	.964	.962
	WIL	.848	.882	.872	.998	.984	.994
α = .10	VDW	.934	.918	.928	1.000	.998	.998
	SIGN	.836	.910	.878	.978	.984	.984
	WIL	.930	.930	.934	1.000	.998	1.000
		θ = 2.00			θ = 2.50		
α = .05	VDW	1.000	.998	.998	1.000	1.000	1.000
	SIGN	.994	.990	.990	1.000	.998	.998
	WIL	1.000	1.000	1.000	1.000	1.000	1.000
α = .10	VDW	1.000	1.000	1.000	1.000	1.000	1.000
	SIGN	.996	.994	.994	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000
		θ = 3.00			θ = 4.00		
α = .05	VDW	1.000	1.000	1.000	1.000	1.000	1.000
	SIGN	1.000	.998	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000
α = .10	VDW	1.000	1.000	1.000	1.000	1.000	1.000
	SIGN	1.000	1.000	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000

(500)

Table III

Sample Size = 15

		NORM	DE	LOG	NORM	DE	LOG
		θ = .00			θ = .25		
α = .05	VDW	.048	.048	.048	.242	.290	.254
	SIGN	.054	.054	.054	.180	.282	.202
	WIL	.052	.052	.052	.226	.290	.248
α = .10	VDW	.096	.096	.096	.358	.404	.374
	SIGN	.092	.092	.092	.274	.412	.320
	WIL	.102	.102	.102	.338	.414	.368
		θ = .50			θ = .75		
α = .05	VDW	.536	.600	.564	.834	.842	.840
	SIGN	.386	.588	.444	.686	.838	.776
	WIL	.534	.624	.578	.822	.870	.852
α = .10	VDW	.686	.736	.696	.922	.922	.924
	SIGN	.526	.720	.592	.796	.920	.864
	WIL	.664	.772	.702	.922	.938	.928
		θ = 1.00			θ = 1.50		
α = .05	VDW	.984	.962	.964	1.000	.998	1.000
	SIGN	.882	.950	.922	.998	1.000	1.000
	WIL	.968	.968	.978	1.000	1.000	1.000
α = .10	VDW	.996	.986	.994	1.000	1.000	1.000
	SIGN	.942	.978	.962	1.000	1.000	1.000
	WIL	.996	.994	.996	1.000	1.000	1.000
		θ = 2.00			θ = 2.50		
α = .05	VDW	1.000	1.000	1.000	1.000	1.000	1.000
	SIGN	1.000	1.000	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000
α = .10	VDW	1.000	1.000	1.000	1.000	1.000	1.000
	SIGN	1.000	1.000	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000
		θ = 3.00			θ = 4.00		
α = .05	VDW	1.000	1.000	1.000	1.000	1.000	1.000
	SIGN	1.000	1.000	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000
α = .10	VDW	1.000	1.000	1.000	1.000	1.000	1.000
	SIGN	1.000	1.000	1.000	1.000	1.000	1.000
	WIL	1.000	1.000	1.000	1.000	1.000	1.000

(500)

TABLES IV and V

TABLE IV

Critical Values

Below are presented critical values for the one sample van der Waerden, sign, and Wilcoxon one sample one sided upper tail tests. It is necessary that a randomized procedure be carried to obtain the exact size stated.

α is the significance level (size). n is the sample size. CV is the critical value. p is the probability of rejection if the test statistic falls precisely on the critical value.

For the van der Waerden test the ranks giving the critical value are also given. For this test, with n = 15, CV is only approximate; see the appendix for details (p.31).

van der Waerden Test (W_n)

	α=.10			α=.05		
n	RANKS	CV	p	RANKS	CV	p
2	(12)	1.3981489	.40	(12)	1.3981489	.20
4	(234)	2.6475733	.60	(1234)	2.9009204	.80
6	(12356)	3.6448719	.40	(3456)	3.8903917	.20
8	(123678)	4.6339344	.60	(234678)	5.0836799	.80
10	(13456789)	5.6286132	.40	(4578910)	6.1084352	.20
15	*	8.0502	1.00	*	8.6668	1.00

Sign Test (S_n)

	α=.10		α=.05	
n	CV	p	CV	p
2	2	.400000	2	.200000
4	3	.150000	4	.800000
6	5	.899787	5	.366738
8	6	.592864	6	.135407
10	7	.386519	8	.893182
15	10	.444929	11	.778846

Wilcoxon Test (W_n)

	α=.10		α=.05	
n	CV	p	CV	p
2	3	.400000	3	.200000
4	9	.611111	10	.806452
6	17	.709677	18	.096774
8	27	.074074	30	.687500
10	40	.157895	44	.727273
15	83	.600000	89	.500000

TABLE V

$$\psi_n(k) = \Phi^{-1}\left(\tfrac{1}{2} + \frac{k}{2(n+1)}\right)$$

where Φ^{-1} is the inverse of Φ, the standard normal distribution function.

$n = 2$

k	$\psi_n(k)$
1	.4307273
2	.9674216

$n = 4$

k	$\psi_n(k)$
1	.2533471
2	.5244005
3	.8416212
4	1.2815516

$n = 6$

k	$\psi_n(k)$
1	.1800124
2	.3661064
3	.5659488
4	.7916386
5	1.0675705
6	1.4652338

$n = 8$

k	$\psi_n(k)$
1	.1397103
2	.2822161
3	.4307273
4	.5894558
5	.7647097
6	.9674216
7	1.2206403
8	1.5932188

$n = 10$

k	$\psi_n(k)$
1	.1141853
2	.2298841
3	.3487557
4	.4727891
5	.6045853
6	.7478586
7	.9084579
8	1.0968036
9	1.3351777
10	1.6906216

$n = 15$

k	$\psi_n(k)$
1	.0784124
2	.1573107
3	.2372021
4	.3186394
5	.4022501
6	.4887764
7	.5791322
8	.6744898
9	.7764218
10	.8871466
11	1.0099902
12	1.1503494
13	1.3180109
14	1.5341205
15	1.8627319

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